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The Ground State and the Long-Time Evolution in the CMC Einstein Flow

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Abstract. Let $(g, K)(k)$ be a CMC (vacuum) Einstein flow over a compact three-manifold Σ with non-positive Yamabe invariant $(Y(\Sigma))$. As noted by Fischer and Moncrief, the reduced volume $\mathcal{V}(k) = \left(\frac{-k}{3}\right)^3 \mathrm{Vol}_{g(k)}(\Sigma)$ is monotonically decreasing in the expanding direction and bounded below by $\mathcal{V}_{\mathrm{inf}} = \left(\frac{-1}{6} Y(\Sigma)\right)^{\frac{3}{2}}$. Inspired by this fact we define the ground state of the manifold Σ as "the limit" of any sequence of CMC states $\{(g_i, K_i)\}$ satisfying: (i) $k_i = -3$, (ii) $\mathcal{V}_i \downarrow \mathcal{V}_{\mathrm{inf}}$, (iii) $Q_0((g_i, K_i)) \leq \Lambda$, where Q_0 is the Bel–Robinson energy and Λ is any arbitrary positive constant. We prove that (as a geometric state) the ground state is equivalent to the Thurston geometrization of Σ. Ground states classify naturally into three types. We provide examples for each class, including a new ground state (the Double Cusp) that we analyze in detail. Finally, consider a long time and cosmologically normalized flow $(\tilde{g}, \tilde{K})(\sigma) = \left(\left(\frac{-k}{3}\right)^2 g, \left(\frac{-k}{3}\right) K\right)$, where $\sigma = -\ln(-k) \in [a, \infty)$. We prove that if $\tilde{\mathcal{E}_1} = \mathcal{E}_1((\tilde{g}, \tilde{K})) \leq \Lambda$ (where $\mathcal{E}_1 = Q_0 + Q_1$, is the sum of the zero and first order Bel–Robinson energies) the flow $(\tilde{g}, \tilde{K})(\sigma)$ persistently geometrizes the three-manifold Σ and the geometrization is the ground state if $\mathcal{V} \downarrow \mathcal{V}_{\mathrm{inf}}$.

1. Introduction

Consider a cosmological space–time solution $\mathbf{g}$ over $\mathbf{M} = \Sigma \times (\sigma_0, \infty)$, where Σ is a compact three-manifold having non-positive Yamabe invariant $Y(\Sigma)$.[1] Suppose that the foliation $\{\Sigma \times \{\sigma\}\}$ is CMC and that σ is the logarithmic time, namely suppose that each slice $\Sigma \times \{\sigma\}$ is of constant mean curvature $k = -e^{-\sigma}$. Consider

This work was completed while the author was a Moore Instructor at MIT.

[1] The Yamabe invariant (sometimes called *sigma constant*) is defined as the supremum of the scalar curvatures of unit volume Yamabe metrics. A Yamabe metric is a metric minimizing the Yamabe functional in a given conformal class.

the Einstein (CMC) flow $(g, K)(\sigma)$ where $g(\sigma)$ and $K(\sigma)$ are the induced three-metric and second fundamental form over each slice $\Sigma\times\{\sigma\}$. A natural question to ask is the following. Suppose we observe the evolution of (g, K) at the cosmological scale, then, is the long-time fate of (g, K) (at the cosmological scale) unique, and if so, how can one characterize it? If the answer is yes, one would naturally call the limit *the ground state* (at the cosmological scale) as any solution would decay to it. In this article we will present partial answers to this question. We elaborate on that below.

It is a simple but interesting fact that (with generality) one can interpret $\frac{-k}{3}$ as equal to the Hubble parameter $\mathcal{H}$ of the "universe" $(\mathbf{g}, \mathbf{M})$ at the "instant of time" $\Sigma \times \{\sigma(k)\}$[14]. This cosmological interpretation of the mean curvature k (or better of $\frac{-k}{3}$) motivates the terminology of various notions that we describe in what follows. Consider a CMC slice $\Sigma \times \{\sigma\}$. At that slice the Hubble parameter is thus $\mathcal{H} = \frac{e^{-\sigma}}{3}$. For this particular value of $\mathcal{H}$ scale $\mathbf{g}$ as $\mathcal{H}^2\mathbf{g}$. As it is easy to see, the state (g, K) over the slice $\Sigma\times\{\sigma\}$ scales to the new state $(\tilde{g}, \tilde{K}) = (\mathcal{H}^2 g, \mathcal{H}K)$. In this way the Hubble parameter of the new solution $\mathcal{H}^2\mathbf{g}$ and over the same slice will be equal to one. A state $(\tilde{g}, \tilde{K})$ with $\mathcal{H} = 1$ (or $k = -3$) will be called a *cosmologically normalized state*. The flow $(\tilde{g}, \tilde{K})(\sigma) = (\mathcal{H}^2(\sigma)g(\sigma), \mathcal{H}(\sigma)K(\sigma))$ will be called the *cosmologically normalized Einstein CMC flow*[2] Note that the volume of Σ relative to the metric $\tilde{g}$ is given by $\mathcal{V}(\sigma) = \mathcal{H}^3(\sigma)\mathrm{Vol}_{g(\sigma)}(\Sigma)$. We will call it the *reduced volume*. It is a crucial and central fact observed by Fischer and Moncrief [8] that $\mathcal{V}$ is monotonically decreasing along the expanding direction and it is bounded below by the topological invariant $\left(-\frac{1}{6}Y(\Sigma)\right)^{\frac{3}{2}}$. The reduced volume is a weak quantity but its relevance is greatly enhanced if we take into account at the same time the $L^2_{\tilde{g}}$ norm of the space–time curvature $\mathbf{Rm}$ relative to the CMC slices, namely the Bel–Robinson energy $\tilde{Q}_0 = Q_0((\tilde{g}, \tilde{K}))$. Our first result in Sect. 2 will be to show that, assuming a uniform bound in $\tilde{Q}_0$, the ground state of the manifold Σ is well defined and unique. In a geometric sense the ground state is equivalent to the Thurston geometrization of Σ. Let us be more precise on the definition of ground state (under a bound in Q_0) and its characterization. By *ground state* we mean "the limit" (to be described below) of any sequence of cosmologically normalized states $\{(\tilde{g}_i, \tilde{K}_i)\}$ with $Q_0((\tilde{g}_i, \tilde{K}_i)) \leq \Lambda$ (Λ is a positive constant) and $\mathcal{V}_i \downarrow \mathcal{V}_{\inf}$. As is shown in the appendix, for any CMC state (g, K) the L^2_g-norm of Ric is controlled by $|k|$, Q_0 and $\mathcal{V}$ and precisely by

$$\|\mathrm{Ric}\|^2_{L^2_g} \leq C(|k|\mathcal{V} + Q_0),$$

where C is a numeric constant. It follows that the Ricci curvature of the sequence $\{\tilde{g}_i\}$ is uniformly bounded in $L^2_{\tilde{g}_i}$. Thus [1], one can extract a subsequence of $\{(\Sigma, \tilde{g}_i)\}$ converging in the weak H^2-topology to a (non-necessarily complete) Riemannian manifold $(\Sigma_\infty, g_\infty)$. We prove that the limit space $(\Sigma_\infty, g_\infty)$ belongs

[2] Cosmologically normalized flows have been considered in [5] by Andersson and Moncrief. Note however that the terminology *Cosmologically normalized* has been introduced in [15].

to one among three possibilities (independently of the sequence $\{(g_i, K_i)\}$). In general terms (see Sect. 2.1 for a more elaborate description of the ground state) the three cases are:

1. (Called *Case* $Y(\Sigma) = 0$), $\Sigma_\infty = \emptyset$;
2. (Called *Case* $Y(\Sigma) < 0$ *(I))*, $\Sigma_\infty = H$ is a hyperbolic manifold and $g_\infty = g_H$ (where g_H is the hyperbolic metric in Σ_∞);
3. (Called *Case* $Y(\Sigma) < 0$ *(II))*, $\Sigma_\infty = \cup_{i=1}^{i=n} H_i$ where $\{H_i\}$ is a finite set of (non-compact) complete hyperbolic metrics of finite volume. The limit metric g_∞ over each H_i is equal to $g_{H,i}$ (where $g_{H,i}$ is the hyperbolic metric of H_i). The two-tori transversal to the hyperbolic cusps of each manifold H_i embed uniquely (up to isotopy) and incompressibly (the π_1 injects) in Σ.

In the second and third cases K_i converges to $-g_{H,i}$ weakly in H^1. One can also describe the notion of ground state in terms of *geometrizations*. This viewpoint will be fundamental in Sect. 3. Recall that for any Riemannian space (Σ, g) the ϵ-thick (thin) part Σ^ϵ (Σ_ϵ) of Σ is defined as the set of points p in Σ where the volume radius[3] $\nu(p)$ is bigger (less) or equal than ϵ. Say now that $\{\tilde{g}(\sigma)\}$ is a continuous ($\sigma \in [\sigma_0, \infty)$) or discrete ($\sigma \in \{\sigma_0, \sigma_1, \ldots\}$) family of Riemannian metrics on Σ. We say that $\{(\Sigma, \tilde{g}(\sigma))\}$ *persistently geometrizes*[4] Σ iff there is $\epsilon(\sigma) > 0$ such that $\Sigma^{\epsilon(\sigma)}_{\tilde{g}(\sigma)}$ is persistently diffeomorphic to either, the empty set, or, the $\epsilon(\sigma)$-thick part of a single compact hyperbolic manifold ($(H, \tilde{g}_H)$), or, the $\epsilon(\sigma)$-thick part of a finite set of (non-compact) complete hyperbolic metrics of finite volume ($\cup_{i=1}^{i=n}(H_i, \tilde{g}_{H,i})$). The $\epsilon(\sigma)$-thin parts $\Sigma_{\tilde{g}(\sigma),\epsilon(\sigma)}$ on the other hand are persistently diffeomorphic to either, the empty set, or, a single graph manifold (G), or, a finite set of graph manifolds with toric boundaries ($\cup_{i=1}^{i=n} G_i$). In quantitative terms $\{\tilde{g}(\sigma)\}$ geometrizes Σ iff either

1. $\overline{\nu}_{\tilde{g}(\sigma)}(\Sigma) \to 0$ as σ goes to infinity (in which case there is only one persistent G piece) or
2. $\underline{\nu}_{\tilde{g}(\sigma)}(\Sigma) \geq \nu_0 > 0$ as σ goes to infinity (in which case there is only one persistent H piece) and there is a continuous function $\varphi : (\sigma_0, \infty) \times H \to \Sigma$, differentiable in the second factor, such that $\|\varphi^* \tilde{g}(\sigma) - \tilde{g}_H\|_{H^2_{\tilde{g}_H}} \to 0$ as σ goes to infinity, or
3. the volume radius collapses in some regions and remains bounded below in some others (in which case there are a set of G pieces $G_1, \ldots, G_j$ and a set of H pieces $H_1, \ldots, H_k$) and for any $\epsilon > 0$ and for any H piece $(H_i, \tilde{g}_{Hi})$ there is a continuous function $\varphi_i : (\sigma_0, \infty) \times H_i^\epsilon \to \Sigma$, differentiable in the second factor such that $\|\varphi_i^* \tilde{g}(\sigma) - \tilde{g}_{Hi}\|_{H^2_{\tilde{g}_{Hi}}} \to 0$ as σ goes to infinity.

It is clear that cases 1, 2 and 3 above correspond, respectively, to the three possible cases (1, 2 and 3) of ground states defined before.

[3] Given a point p in Σ the volume radius $\nu(p)$ at p is defined as the supremum of all $r > 0$ such that $\mathrm{Vol}(B(p, r)) \geq \mu r^3$ for some fixed (but arbitrary) $\mu > 0$. We define $\underline{\nu} = \inf_{p \in \Sigma} \nu(p)$ and $\overline{\nu} = \sup_{p \in \Sigma} \nu(p)$. We will be using these definitions later.

[4] We have taken this terminology from [11, Sect. 10].

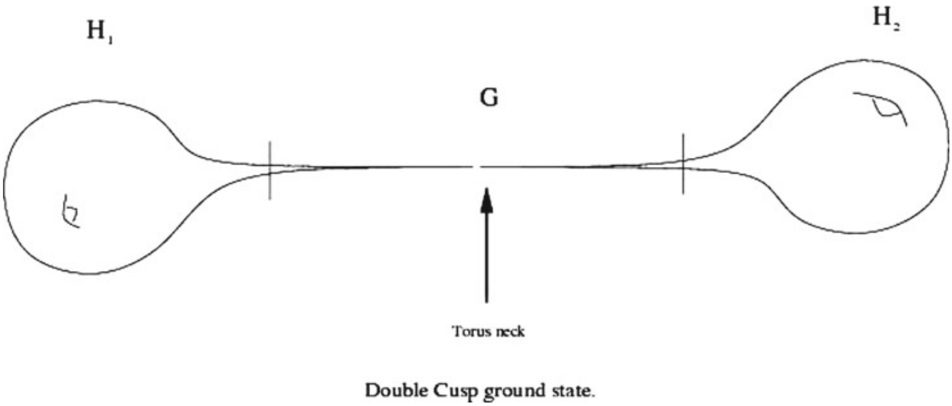

Figure 1. A schematic representation of the double cusp ground state.

While it is easy to give examples of ground states of the type *Case* $Y(\Sigma) = 0$ and *Case* $Y(\Sigma) < 0$ *(I)* (see Sect. 2.2) an example of the type *Case* $Y(\Sigma) < 0$ *(II)* is more difficult to find. We dedicate Sect. 6 to describe a ground state of this type. The new ground state, that we shall call *Double Cusp*, consists of a family $\{\Sigma, (\tilde{g}_l, \tilde{K}_l)\}$ that we describe in what follows. The manifold Σ is of the form $\Sigma = H_1 \sharp G \sharp H_2$ where H_i, $i = 1, 2$ are (non-compact) hyperbolic manifolds with a hyperbolic cusp each[5] and the manifold G is a so called *torus neck* $G = [-1, 1] \times T^2$. The family $\{(\tilde{g}_l, \tilde{K}_l)\}$ is parametrized by the metric "length" l of the neck. As $l \to \infty$ the geometrization takes place. More precisely, as the length l of G becomes infinite, the volume radius $\overline{\nu}(G)$ over G and the total volume of G collapse to zero. Over the hyperbolic sector H_1, H_2 instead, the metric g_l converges to g_{H_1} and g_{H_2} respectively and in H^2. A schematic picture can be seen in Fig. 1.

The third part of the article (Sect. 3) deals with the long-time evolution of the cosmologically normalized Einstein flow under the assumption that the zero and the first order Bel–Robinson energies remain uniformly bounded, namely $\tilde{\mathcal{E}}_1 = Q_0((\tilde{g}, \tilde{K})) + Q_1((\tilde{g}, \tilde{K})) \leq \Lambda$ for a positive constant Λ. The main result will be to show that a long-time flow $(\tilde{g}, \tilde{K})(\sigma)$ with $\tilde{\mathcal{E}}_1 \leq \Lambda$, persistently geometrizes the manifold Σ. Moreover, the geometrization is the ground state if $\mathcal{V} \downarrow \mathcal{V}_{inf}$. Using the classification of ground states (Theorem 3) it is direct to show that ground states are stable in the following sense. For any Λ there is $\epsilon > 0$ such that any long-time cosmologically normalized flow $(\tilde{g}, \tilde{K})(\sigma)$ with $\tilde{\mathcal{E}}_1 \leq \Lambda$ and initial data $(\tilde{g}(\sigma_0, \tilde{K}(\sigma_0))$ with $\mathcal{V}(\sigma_0) - \mathcal{V}_{inf} \leq \epsilon$ the flow converges to the ground state in the long time (in the sense of geometrizations). This result is not known in general if one drops the a priori (strong) assumption of a uniform bound on $\tilde{\mathcal{E}}_1$. However, it was proved by Andersson and Moncrief [5] that if an initial data $(\tilde{g}(\sigma_0), \tilde{K}(\sigma_0))$ is

[5] The construction can be easily generalized to include hyperbolic manifolds with any number of cusps.

close enough in $H^3 \times H^2$ to a ground state $(H, (g_H, -g_H))$ of type *Case* $Y(\Sigma) < 0$ *(I)*, then $\tilde{\mathcal{E}}_1$ converges to zero when $\sigma \to \infty$ and $\mathcal{V} \downarrow \mathcal{V}_{inf}$, thus showing stability. We will give a proof of this fact in slightly more geometrical terms. The core of the proof is however the same.

Finally, in Sect. 4 we present some arguments favoring the statement that a cosmologically scaled long-time CMC flow with $\tilde{\mathcal{E}}_1$ uniformly bounded decays necessarily to its ground state.

The problem of the long-time geometrization of the Einstein flow was investigated by Anderson in the seminal work [4]. In that article, long-time geometrization (at a particular scale) was established under suitable a priori point-wise bounds on the rescaled space–time curvature (see [4] for a detailed statement). More precisely, it was shown that the flow geometrizes along (some) sequence of diverging times. The problem of whether the geometrization persists or not, and a careful analysis of the collapsed regions remained open. Further progress on these problems was done in [15], where it was possible to show that under a priori C^α point-wise bounds on the cosmological normalized space–time curvature, the flow (at the cosmological scale) persistently geometrizes and the volume of the G (or thin) regions decreases to zero. An important open problem is whether the two-tori, separating the H (or thick) regions and G (or thin) regions are incompressible or not. A positive answer would imply that the geometrization is unique, that coincides with the Thurston geometrization and that the reduced volume approaches its absolute infimum in the long-time (see [15] for a discussion). The present article discusses the problem of the long-time geometrization under, instead, a priori integral bounds on the cosmologically normalized space–time curvature and its time derivative. These integral norms (the Bel–Robinson energies $\tilde{Q}_0$ and $\tilde{Q}_1$) represent variables which go more in the spirit of general relativity, if we think (at least formally) the Einstein equations as a hyperbolic PDE. The analysis of the Einstein flow under a priori bounds on $\tilde{Q}_0 + \tilde{Q}_1$ is, in comparison with point-wise bounds on $\tilde{\mathbf{Rm}}$ and $\boldsymbol{\nabla}_T\tilde{\mathbf{Rm}}$, a task of much greater complexity. Finally let us note that the idea of linking the reduced volume to geometrizations was first pointed out and investigated in a series of articles by Fischer and Moncrief (see for instance [8]). The reference [15] and the present work owe much to them.

1.1. Background

We summarize now some basic formulae that will be used. The reader is encouraged to read [7] from which most of the material of this section is taken (see also [16] for related material). Let us assume we have a cosmological solution[6] $(\mathbf{M}, \mathbf{g})$ with generic Cauchy hypersurface diffeomorphic to Σ. Assume Σ is a compact three-manifold with non-positive Yamabe invariant $Y(\Sigma)$. Assume too that there is a CMC foliation $\Sigma \times [k_0, 0)$ inside[7] $\mathbf{M}$, where k is the mean curvature. A solution

[6] Following Bartnik a *cosmological solution* of the Einstein equations is a maximally globally hyperbolic solution having a compact space-like Cauchy hypersurface.

[7] Not necessarily covering the future of $\mathbf{M}$.

having such foliation will be called a *long-time CMC solution*.[8] With respect to the CMC foliation the metric $\mathbf{g}$ splits into a space-like metric g, a lapse N and a shift X. We recover the metric $\mathbf{g}$ from them by

$$\mathbf{g} = -(N^2 - |X|^2)\mathrm{d}k^2 + X^* \otimes \mathrm{d}k + \mathrm{d}k \otimes X^* + g,$$

where $X^* = g_{ab}X^a$. The Einstein CMC equations in the CMC gauge (and for an arbitrary shift) are

$$R = |K|^2 - k^2, \tag{1}$$

$$\nabla . K = 0, \tag{2}$$

$$\dot{g} = -2NK + \mathcal{L}_X g, \tag{3}$$

$$\dot{K} = -\nabla\nabla N + N(\mathrm{Ric} + kK - 2K \circ K) + \mathcal{L}_X K, \tag{4}$$

$$-\Delta N + |K|^2 N = 1, \tag{5}$$

where K is the second fundamental form, $(K \circ K)_{ab} = K_a{}^c K_{cb}$ and $\mathcal{L}_X$ is the Lie derivative operator along the vector field X. Sometimes we will need to use these formulas in terms of the cosmologically normalized quantities $\tilde{g} = \mathcal{H}^2 g$, $\tilde{K} = \mathcal{H}K$ and $\tilde{N} = \mathcal{H}^2 N$. They will be provided without further deductions.

The expressions for the derivative of the reduced volume with respect to logarithmic time will be central in Sect. 3. It is convenient to write them right away in terms of cosmological normalized quantities. They are

$$\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\sigma} = -3 \int_\Sigma 1 - 3\tilde{N}\mathrm{d}v_{\tilde{g}} = -\int_\Sigma \tilde{N}|\hat{\tilde{K}}|^2_{\tilde{g}}\mathrm{d}v_{\tilde{g}}.$$

where the hat $\hat{A}$ of a two tensor A denotes its traceless part $\hat{A} = A - \frac{tr_g A}{3} g$. The expression $\phi = 3\tilde{N} - 1$ is the so called *Newtonian potential* and it is sometimes a better quantity to work rather than the lapse N.

Let us give now the basic elements of Weyl fields and Bel–Robinson energies. Again in this case the reader is encouraged to read the reference [7] for a complete account. A Weyl field is a traceless $(4, 0)$ space–time tensor field having the symmetries of the curvature tensor $\mathbf{Rm}$. We will denote them by $\mathbf{W}_{abcd}$ or simply $\mathbf{W}$. As an example, the Riemann tensor in a vacuum solution of the Einstein equations is a Weyl field that we will be denoting by $\mathbf{Rm} = \mathbf{W}_0$ (we will use indistinctly either $\mathbf{Rm}$ or $\mathbf{W}_0$). The covariant derivative of a Weyl field $\boldsymbol{\nabla}_X \mathbf{W}$ for an arbitrary vector field X is also a Weyl field. We will be using the Weyl fields $\mathbf{W}_0 = \mathbf{Rm}$ and $\mathbf{W}_1 = \boldsymbol{\nabla}_T \mathbf{Rm}$, where T is the future pointing unit normal field to the CMC foliation.

Given a Weyl tensor $\mathbf{W}$ define the current $\mathbf{J}$ by

$$\boldsymbol{\nabla}^a \mathbf{W}_{abcd} = \mathbf{J}_{bcd},$$

[8] The terminology is justified by the fact that if the manifold Σ has non-positive Yamabe invariant then the range of k (which is known to be a connected interval of the real line) cannot contain zero. If $Y(\Sigma) \leq 0$ it is conjectured that the range of k is actually $(-\infty, 0)$.

When $\mathbf{W}$ is the Riemann tensor in a vacuum solution of the Einstein equations the currents $\mathbf{J}$ is zero due to the Bianchi identities.

The L^2-norm of a Weyl field $\mathbf{W}$ with respect to the foliation will be introduced through the Bel–Robinson tensor which is defined by

$$Q_{abcd}(\mathbf{W}) = \mathbf{W}_{alcm}\mathbf{W}_{b\ d}^{\ l\ m} + \mathbf{W}^*_{alcm}\mathbf{W}^{*\ l\ m}_{\ b\ d}.$$

The Bel–Robinson tensor is symmetric and traceless in all pairs of index and for any pair of time-like vectors T_1 and T_2, the quantity[9] $Q_{abcd}T_1^aT_2^bT_3^cT_4^d$ is positive (provided $\mathbf{W} \neq 0$).

The electric and magnetic components of $\mathbf{W}$ are defined as

$$E_{ab} = \mathbf{W}_{acbd}T^cT^d, \tag{6}$$

$$B_{ab} =^* \mathbf{W}_{acbd}T^cT^d, \tag{7}$$

where the left dual of $\mathbf{W}$ is defined by ${}^*\mathbf{W}_{abcd} = \frac{1}{2}\epsilon_{ablm}\mathbf{W}^{lm}_{\ \ cd}$. E and B are symmetric, traceless and null in the T direction. It is also the case that $\mathbf{W}$ can be reconstructed from them (see [p. 143] [7]). If $\mathbf{W}$ is the Riemann tensor in a vacuum solution we have

$$E_{ab} = \mathrm{Ric}_{ab} + kK_{ab} - K_a^{\ c}K^c_{\ b}, \tag{8}$$

$$\epsilon_{ab}^{\ \ l}B_{lc} = \nabla_a K_{bc} - \nabla_b K_{ac}. \tag{9}$$

The components of a Weyl field with respect to the CMC foliation are given by $(i, j, k, l$ are spatial indices)

$$\mathbf{W}_{ijkT} = -\epsilon_{ij}^{\ \ m}B_{mk}, \quad {}^*\mathbf{W}_{ijkT} = \epsilon_{ij}^{\ \ m}E_{mk}, \tag{10}$$

$$\mathbf{W}_{ijkl} = \epsilon_{ijm}\epsilon_{kln}E^{mn}, \quad {}^*\mathbf{W}_{ijkl} = \epsilon_{ijm}\epsilon_{kln}B^{mn}. \tag{11}$$

We also have

$$Q_{TTTT} = |E|^2 + |B|^2,$$
$$Q_{iTTT} = 2(E \wedge B)_i,$$
$$Q_{ijTT} = -(E \times E)_{ij} - (B \times B)_{ij} + \frac{1}{3}(|E|^2 + |B|^2)g_{ij}.$$

The operations $\times$ and $\wedge$ are provided explicitly later. The divergence of the Bel–Robinson tensor is

$$\begin{aligned}\boldsymbol{\nabla}^a Q(\mathbf{W})_{abcd} &= \mathbf{W}_{b\ d}^{\ m\ n}\mathbf{J}(\mathbf{W})_{mcn} + \mathbf{W}_{b\ c}^{\ m\ n}\mathbf{J}(\mathbf{W})_{mdn} \\ &\quad +{}^*\mathbf{W}_{b\ d}^{m\ n}\mathbf{J}^*(\mathbf{W})_{mcn} +{}^* \mathbf{W}_{b\ c}^{m\ n}\mathbf{J}^*(W)_{mcn}.\end{aligned}$$

where $\mathbf{J}^*_{bcd} = \boldsymbol{\nabla}^a({}^*\mathbf{W}_{abcd})$. We have therefore

$$\boldsymbol{\nabla}^\alpha Q(\mathbf{W})_{\alpha TTT} = 2E^{ij}(\mathbf{W})\mathbf{J}(\mathbf{W})_{iTj} + 2B^{ij}\mathbf{J}^*(\mathbf{W})_{iTj}. \tag{12}$$

We will denote

$$Q(\mathbf{W}) = \int_\Sigma Q_{TTTT}(\mathbf{W})\mathrm{d}v_g = \int_\Sigma |E(\mathbf{W})|^2 + |B(\mathbf{W})|^2\mathrm{d}v_g,$$

[9] We will later use the notation $Q_{abcd}T_1^aT_2^bT_3^cT_4^d = Q_{T_1T_2T_3T_4}$.

and in particular when $\mathbf{W} = \mathbf{W}_0$ or $\mathbf{W} = \mathbf{W}_1$ we will denote $Q_0 = Q(\mathbf{W}_0)$ and $Q_1 = Q(\mathbf{W}_1)$. From the Eq. (12) we get the *Gauss equation*

$$\dot{Q}(\mathbf{W}) = -\int_{\Sigma} 2NE^{ij}(\mathbf{W})\mathbf{J}(\mathbf{W})_{iTj} + 2NB^{ij}(\mathbf{W})\mathbf{J}^*(\mathbf{W})_{iTj} + 3NQ_{abTT}\mathbf{\Pi}^{ab}\mathrm{d}v_g. \tag{13}$$

$\mathbf{\Pi}_{ab} = \mathbf{\nabla}_a T_b$ is the *deformation tensor* and plays a fundamental role in the space–time tensor algebra. Its components are

$$\mathbf{\Pi}_{ij} = -K_{ij}, \quad \mathbf{\Pi}_{iT} = 0,$$
$$\mathbf{\Pi}_{Ti} = \frac{\nabla_i N}{N}, \quad \mathbf{\Pi}_{TT} = 0.$$

Finally, we have

$$\mathrm{div}E(\mathbf{W})_a = (K \wedge B(\mathbf{W}))_a + \mathbf{J}_{TaT}(\mathbf{W}), \tag{14}$$
$$\mathrm{div}B(\mathbf{W})_a = -(K \wedge E(\mathbf{W}))_a + \mathbf{J}^*_{TaT}(\mathbf{W}), \tag{15}$$
$$\mathrm{curl}B_{ab}(\mathbf{W}) = E(\mathbf{\nabla}_T\mathbf{W})_{ab} + \frac{3}{2}(E(\mathbf{W}) \times K)_{ab} - \frac{1}{2}kE_{ab}(\mathbf{W}) + \mathbf{J}_{aTb}(\mathbf{W}), \tag{16}$$
$$\mathrm{curl}E_{ab}(\mathbf{W}) = B(\mathbf{\nabla}_T\mathbf{W})_{ab} + \frac{3}{2}(B(\mathbf{W}) \times K)_{ab} - \frac{1}{2}kB_{ab}(\mathbf{W}) + \mathbf{J}^*_{aTb}(\mathbf{W}). \tag{17}$$

The operations $\wedge$, $\times$ and the operators Div and Curl are defined through

$$(A \times B)_{ab} = \epsilon_a{}^{cd}\epsilon_b{}^{ef}A_{ce}B_{df} + \frac{1}{3}(A \circ B)g_{ab} - \frac{1}{3}(trA)(trB)g_{ab},$$
$$(A \wedge B)_a = \epsilon_a{}^{bc}A_b{}^d B_{dc},$$
$$(\mathrm{div}\ A)_a = \nabla_b A^b{}_a,$$
$$(\mathrm{curl}\ A)_{ab} = \frac{1}{2}(\epsilon_a{}^{lm}\nabla_l A_{mb} + \epsilon_b{}^{lm}\nabla_l A_{ma}).$$

In what follows we describe the main results that will be used from the theory of convergence–collapse of Riemannian manifolds under L^2-bounds on the Ricci curvature and its covariant derivatives. The reader can consult (some of) the original sources [1,17,18]. Let us mention first a classical local result. Recall that in a Riemannian manifold (Σ, g) the *H^i-harmonic radius* $r_i(p)$ of g at p, $i \geq 2$, is defined as the supremum of the radius r for which there is a coordinate chart $\{x\}$ covering $B(p,r)$ and satisfying

$$\frac{3}{4}\delta_{jk} \leq g_{jk} \leq \frac{4}{3}\delta_{jk}, \tag{18}$$

$$\sum_{\alpha=2}^{\alpha=i} r^{2\alpha-3}\left(\sum_{|I|=\alpha,j,k}\int_{B(o,r)} \left|\frac{\partial^I}{\partial x^I}g_{jk}\right|^2 \mathrm{d}v_x\right) \leq 1, \tag{19}$$

where in the sum above I is the multi-index $I = (\alpha_1, \alpha_2, \alpha_3)$, and as usual $\partial^I/\partial x^I = (\partial_{x^1})^{\alpha_1}(\partial_{x^2})^{\alpha_2}(\partial_{x^3})^{\alpha_3}$. Both expressions above are invariant under the simultaneous scaling $\tilde{g} = \lambda^2 g$, $\tilde{x}^\mu = \lambda x^\mu$ and $\tilde{r} = \lambda r$. Observe that if $j > i \geq 2$

then $r_j(p) \leq r_i(p)$. A chart $\{x\}$ as above will be called a *canonical harmonic chart*. In basic terms the H^i-harmonic radius marks the greatest scale λ in which the $H^i_{\{x\}}$-Sobolev norm of the scaled metric $\lambda^2 g$ (in the ball of radius one of the scaled metric) is bounded above by one.

We have used the notation $H^i_{\{x\}}$ for the H^i-Sobolev space defined with respect to a chart $\{x\}$. Instead H^i_g will be the H^i_g-Sobolev space defined with respect to the metric g, namely for a tensor A (of any rank) we define

$$\|A\|^2_{H^i_g(\Omega)} = \int_\Omega \sum_{j=0}^{j=i} |\nabla^{(j)} A|^2 \mathrm{d}v_g,$$

where Ω is a given region in Σ where the Sobolev space is defined (so properly speaking we would write $H^i_g(\Omega)$).

Theorem 1. *Let (Σ, g) be a complete Riemannian three-manifold and let p be a point in Σ. Then*

1. $\|\mathrm{Ric}\|^2_{L^2_g}$ *and $\nu(p)$ control the H^2-harmonic radius $r_2(p)$ of g at p from below.*
2. $\|\mathrm{Ric}\|^2_{H^1_g}$ *and $\nu(p)$ control the H^3-harmonic radius $r_3(p)$ of g at p from below.*

A more complex global result is[10]

Theorem 2. *Let $\{(\Sigma, g_i)\}$ be a sequence of compact Riemannian manifolds with*

$$\|\mathrm{Ric}\|_{L^2_{g_i}} + Vol_{g_i}(\Sigma) \leq \Lambda,$$

where Λ is a positive constant. Then one can extract a sub-sequence (to be denoted also by $\{(\Sigma, g_i)\}$) with one of the following behaviors.

(1) *(Collapse). $\overline{\nu}_i \to 0$ and the sub-sequence g_i collapses along a sequence of F-structures. The manifold Σ is in this case a graph manifold.*

(2) *(Convergence). $\underline{\nu}_i \geq \underline{\nu}_0 > 0$ and $\{(\Sigma, \{g_i\})\}$ converges weakly in H^2 to a H^2 Riemannian manifold $(\Sigma_\infty (= \Sigma), g_\infty)$.*

(3) *(Convergence–Collapse). $\underline{\nu}_i \to 0$ and $\overline{\nu}_i \geq \overline{\nu}_0 > 0$ and $\{(\Sigma, g_i)\}$ converges weakly in H^2 to a (at most) countable union $\cup_\alpha (\Sigma_{\infty,\alpha}, g_{\infty,\alpha})$ of H^2 (non necessarily complete) Riemannian manifolds. Moreover, for a given ϵ (sufficiently small) the manifolds $\Sigma_{g_i,\epsilon}$ are graph manifolds with toric boundaries. The Riemannian-manifolds $\{(\Sigma^\epsilon_{g_i}, g_i)\}$ converge weakly in H^2 to $\cup_\alpha (\Sigma^\epsilon_{\infty,\alpha}, g_{\infty,\alpha})$ (which has only a finite number of components).*

The notion of convergence that we have assumed in the statement of the theorem is the following: we say that $\{(\Sigma, g_i)\}$ converges weakly in H^2 to a limit Riemannian manifold $(\Sigma_\infty, g_\infty)$ (as above) if for every $\epsilon > 0$ there are (H^3)-diffeomorphisms $\varphi_i : \Sigma^\epsilon_{\infty,g_\infty} \to \Sigma^\epsilon_{g_i}$ such that $\varphi^*_i g_i$ converges to g_∞ in the weak

[10] For a discussion of this result see [1]. We will not elaborate on the notion of $\mathcal{F}$ structure as we will not need in the present article. A graph manifold is, roughly speaking a sum along two tori, of $U(1)$-bundles over two-surfaces. For a discussion of the relation between graph manifolds and $\mathcal{F}$ structures see [1].

H^2-topology induced by the metric g_∞ over the space of H^2 (2,0)-tensors (over Σ_∞).

The use of Weyl fields to control the gravitational field would not be justified in this article if it were not for the following fundamental property.

Proposition 1. *Let (g, K) be a cosmologically normalized state on a three-manifold Σ with non-negative Yamabe invariant $(Y(\Sigma) \leq 0)$ and let p be a point in Σ. Then*

1. *$\nu(p)$, Q_0 and $Vol_g(\Sigma)$ control from below the H^2-harmonic radius $r_2(p)$ of g at p. Moreover if $\{x\}$ is a canonical harmonic coordinate system the Sobolev norm $\|K\|_{H^1_{\{x\}}(B(p,r_2(p))}$ is controlled from above.*
2. *$\nu(p)$, $Q_0 + Q_1$ and $Vol_g(\Sigma)$ control from below the H^3-harmonic radius $r_3(p)$ of g at p. Moreover if $\{x\}$ is a canonical harmonic coordinate system the Sobolev norm $\|K\|_{H^2_{\{x\}}(B(p,r_3(p))}$ is controlled from above.*

In basic words what the (say *item 2* of the) Proposition claims is that the quantity $1/\nu(p) + Q_0 + Q_1 + \mathrm{Vol}_g(\Sigma)$ controls $r_3(p)$ from below and the $H^2_{\{x\}} \times H^1_{\{x\}}$ norm of (g, K) from above, in suitable canonical harmonic coordinates $\{x\}$ covering $B(p, r_3(p))$.

The proof of Proposition 1 *item 1* follows easily from the bound

$$\left(\int_\Sigma |\mathrm{Ric}|^2 + |\nabla K|^2 + |K|^4 \mathrm{d}v_g\right)^{\frac{1}{2}} \leq C(\mathrm{Vol}_g(\Sigma) + Q_0),$$

that is proved in the appendix (C is a numeric constant) and by applying Theorem 1. The proof of the *item* 2 in the Proposition 1 follows by using *item* 1 and applying standard elliptic estimates in the elliptic system (14)–(17) with $E = E(\mathbf{W}_0)$ and $B = B(\mathbf{W}_0)$ on the left-hand side and then using Eqs. (8) and (9). Details of this argument can be found in [16]. The estimates of Proposition 1 will be referred later simply as "*elliptic estimates*".

2. The Ground State and Examples

2.1. The Ground State

Theorem 3. (The ground state) *Let Σ be a compact three-manifold with $Y(\Sigma) \leq 0$. Say $\{(g_i, K_i)\}$ is a sequence of states satisfying*

$$(1)\quad k_i = -3;$$

$$(2)\quad \mathcal{V}_i \downarrow \mathcal{V}_{inf} = \left(-\frac{1}{6}Y(\Sigma)\right)^{\frac{3}{2}};$$

$$(3)\quad Q_0((g_i, K_i)) \leq \Lambda,$$

where Λ is a fixed constant. Then, there is a sub-sequence of $\{(g_i, K_i)\}$ (to be denoted also by $\{(g_i, K_i)\}$) for which one and only one of the following three phenomena occurs.

Case $Y(\Sigma) = 0$.

1. *$\Sigma = G$ is a graph manifold.*
2. *$\overline{\nu} \to 0$ and the Riemannian spaces (Σ, g_i) collapse with bounded L^2 curvature, along a sequence of F-structures.*
3. *$\mathcal{V}_i \downarrow \mathcal{V}_{inf} = 0$*

Case $Y(\Sigma) < 0$ (I).

1. *$\Sigma = H$ is a compact hyperbolic manifold (denote its hyperbolic metric by g_H).*
2. *$(\Sigma, g_i) \to (\Sigma, g_H)$ in the weak H^2-topology.*
3. *$\mathcal{V}_i \downarrow Vol_{g_H} = \left(-\frac{1}{6}Y(\Sigma)\right)^{\frac{3}{2}}$.*

Case $Y(\Sigma) < 0$ (II).

1. *There is a set of incompressible two-tori $\{T_i^2, i = 1, \ldots, i_T\}$ embedded in Σ and cutting it into a set $\{H_i, i = 1, \ldots, i_H\}$ of manifolds admitting a complete hyperbolic metric of finite volume (in its interior) and a set $\{G_i, i = 1, \ldots, i_G\}$ of graph manifolds. The tori T_i^2 are unique up to isotopy.*
2. *$(\Sigma, g_i) \to \cup_{i=1}^{i=i_H}(H_i, g_{H,i})$ in the weak H^2-topology.*
3. *$\mathcal{V}_i \downarrow \sum_{i=1}^{i=i_H} Vol_{g_{H,i}}(H_i) = \left(-\frac{1}{6}Y(\Sigma)\right)^{\frac{3}{2}}$.*

In each of the three cases above the norms $\|\mathrm{Ric}_{g_i}\|_{L^2_{g_i}}$, $\|K_i\|_{H^1_{g_i}}$, and $\|K_i\|_{L^4_{g_i}}$ remain uniformly bounded and the norms $\|\hat{K}_i\|_{L^2_{g_i}}$, $\|R_{g_i} + 6\|_{L^1_{g_i}}$ converge to zero. Moreover in the regions of convergence (the hyperbolic sector in (I) and (II)) the scalar curvature R_{g_i} converges in the strong L^2-topology to -6.

Finally, two different sub-sequences of the original sequence $\{(g_i, K_i)\}$ as above have the same behavior.

Proof. We will make use of a number of inequalities proved in the Appendix. From Proposition 10 we have

$$\int_\Sigma 2|\nabla \hat{K}|^2 + |\hat{K}|^4 \mathrm{d}v_g \leq C(|k|(\mathcal{V} - \mathcal{V}_{\mathrm{inf}}) + Q_0), \tag{20}$$

and from Proposition 11

$$\int_\Sigma |k|^2 |\hat{K}|^2 \mathrm{d}v_g \leq C(|k|(\mathcal{V} - \mathcal{V}_{\mathrm{inf}}) + (|k|(\mathcal{V} - \mathcal{V}_{\mathrm{inf}})Q_0)^{\frac{1}{2}}). \tag{21}$$

This in particular implies the inequality

$$\int_\Sigma |k|^2 \left(R_g + \frac{2}{3}k^2\right) \mathrm{d}v_g \leq C(|k|(\mathcal{V} - \mathcal{V}_{\mathrm{inf}}) + (|k|(\mathcal{V} - \mathcal{V}_{\mathrm{inf}})Q_0)^{\frac{1}{2}}). \tag{22}$$

From Proposition 12 we have

$$\|\widehat{\mathrm{Ric}}\|_{L^2_g} \leq C\big(|k|(\mathcal{V} - \mathcal{V}_{\mathrm{inf}}) + (|k|(\mathcal{V} - \mathcal{V}_{\mathrm{inf}})Q_0)^{\frac{1}{2}} + Q_0\big). \tag{23}$$

and from Proposition 13 also in the Appendix

$$\int_{\Sigma} |\nabla R|^{\frac{4}{3}} + R^2 \mathrm{d}v_g \leq C(|k|\mathcal{V} + Q_0). \tag{24}$$

Recall that when using the formulae above we will be dealing with a sequence $\{(\tilde{g}_i, \tilde{K}_i)\}$ with $k_i = -3$.

Case $Y(\Sigma) = 0$. From (23) we see that the $L^2_{g_i}$ norm of Ric_{g_i} remains uniformly bounded. This case then follows from Theorem 2.

Case $Y(\Sigma) < 0$. First we note that there must be a constant $\overline{\nu}_0 > 0$ such that $\overline{\nu}_i \geq \overline{\nu}_0$ otherwise one can extract a sub-sequence of $\{g_i\}$ which collapses with bounded volume and curvature. Theorem 2 then implies that Σ is a graph manifold and therefore of zero Yamabe invariant which is a contradiction. Cases (I) and (II) will be distinguished according to whether there is a sub-sequence of $\{g_i\}$ with $\underline{\nu}_i \to 0$ or not. We do that below.

(I). Suppose there exists $\underline{\nu}_0 > 0$ such that $\underline{\nu}_i \geq \underline{\nu}_0$. Then by Theorem 2 we can extract a sub-sequence of $\{g_i\}$ converging in the weak H^2-topology to a compact Riemannian manifold (Σ, g_∞). From (22) we deduce that $R_{g_\infty} = -6$. Let us see that g_∞ is hyperbolic. Note that $\int_\Sigma R^2_{g_\infty} dv_{g_\infty} = |Y(\Sigma)|^2$. Consider the quadratic functional $\mathcal{R}$ from H^2-metrics into the reals given by

$$g \to \mathrm{Vol}_g(\Sigma)^{\frac{1}{3}} \int_\Sigma R_g^2 \mathrm{d}v_g.$$

It is known [2] that if $Y(\Sigma) < 0$ the infimum of $\mathcal{R}$ is given by $|Y(\Sigma)|^2$. Thus it must be $\delta\mathcal{R}|_{g_\infty} = 0$. Let us compute the variation of $\mathcal{R}$ at $g = g_\infty$ and for variations which preserve the local volume. Consider then an arbitrary path of metrics $g(\lambda)$ with $g(\lambda = 0) = g_\infty$ and $(dv_{g(\lambda)})' = 0$ (and Frechet derivative $g' = h$ in H^2). From $(\mathrm{d}v_g)' = 0$ we get $tr_g h = 0$. Recall that the variation of the scalar curvature is given by

$$\delta_h R_g = \Delta tr_g h + \delta\delta h - \langle \mathrm{Ric}, h\rangle.$$

From this we get

$$\delta_h \mathcal{R}_g|_{g=g_\infty} = -\mathrm{Vol}_{g_\infty}^{\frac{1}{3}} 2R_\infty \int_\Sigma \langle \widehat{\mathrm{Ric}}_{g_\infty}, h\rangle \mathrm{d}v_{g_\infty}.$$

Thus, $\widehat{\mathrm{Ric}}_{g_\infty} = 0$ and g_∞ is hyperbolic. Therefore this case corresponds to *Case* $Y(\Sigma) < 0$ (I).

(II). Suppose $\limsup \underline{\nu}_i = 0$. Consider a H^2-weak limit of (Σ, g_i). Denote it by $(\Sigma_\infty, g_\infty)$. Recall that Σ_∞ may have infinitely many connected components and that g_∞ may not be complete on them. Note that Σ_∞ is non-empty as we have $\overline{\nu}_i \geq \overline{\nu}_0 > 0$. For every i consider the metric $g_{Y,i}$ in the conformal class of g_i with scalar curvature $R_Y = -6$. Writing $g_{Y,i} = \phi_i^4 g_i$, the conformal factor ϕ_i satisfies the equation

$$R_Y \phi_i^5 = -8\Delta_{g_i}\phi_i + R_{g_i}\phi_i. \tag{25}$$

From the maximum principle we get $\phi_i \leq 1$. Thus

$$0 \leq \mathrm{Vol}_{g_i}(\Sigma) - \mathrm{Vol}_{g_{Y,i}}(\Sigma) \leq \mathrm{Vol}_{g_i}(\Sigma) - \mathrm{Vol}_{\mathrm{inf}}.$$

It follows from the fact that $\mathcal{V}_i \downarrow \mathcal{V}_{\mathrm{inf}}$ that

$$0 \leq \int\limits_{\Sigma} 1 - \phi_i^6 \mathrm{d}v_{g_i} \to 0.$$

and in particular $\int_\Sigma (1-\phi_i)^6 \mathrm{d}v_{g_i} \to 0$. Note that

$$\mathrm{Vol}^{\frac{1}{3}}_{g_{Y,i}} \int\limits_{\Sigma} R^2_{g_{Y,i}} \mathrm{d}v_{g_{Y,i}} \to |Y(\Sigma)|^2.$$

We will exploit this fact in what follows. Pick an arbitrary point $p \in \Sigma_\infty$. We will show that $\widehat{\mathrm{Ric}}_{g_\infty}|_{B(p,\nu_{g_\infty}(p)/2)} = 0$. As the point p is arbitrary this would show that $\widehat{\mathrm{Ric}}_{g_\infty} = 0$ and thus g_∞ is hyperbolic. First note that by (22) it is $R_{g_\infty} = R_Y = -6$. Also by (24) and the compact embedding $H^{1,4/3} \hookrightarrow L^2$ we see that $R_{g_i} \to R_Y$ strongly in L^2 on compact sets of Σ_∞. Pick a sequence $\{p_i\}$ of points $p_i \in \Sigma$ such that $(B_{g_i}(p_i, \nu_{g_i}(p_i)), p_i, g_i)$ converges to $(B_{g_\infty}(p, \nu_{g_\infty}(p)), p, g_\infty)$. Let us write Eq. (25) in the form

$$8\Delta_{g_i}\phi_i = R_{g_i}\phi_i - R_Y \phi_i^5, \tag{26}$$

and prove that the right hand side of it is converges to zero in L^2 and over the sequence of balls $B_{g_i}(p_i, \nu_{g_i}(p_i))$ (denote them by B_i). Write

$$\begin{aligned}\int\limits_{B_i} |R_Y\phi_i^5 - R_{g_i}\phi_i|^2 \mathrm{d}v_{g_i} &\leq \int\limits_{B_i} |R_Y\phi_i^4 - R_{g_i}|^2 \mathrm{d}v_{g_i} \\ &\leq \int\limits_{B_i} 2|R_Y|^2|\phi^4 - 1|^2 + 2|R_Y - R_{g_i}|^2 \mathrm{d}v_{g_i}.\end{aligned} \tag{27}$$

We have

$$\int\limits_{B_i} |\phi_i^4 - 1|^2 \mathrm{d}v_{g_i} \leq \int\limits_{B_i} |\phi_i^4 - 1| \mathrm{d}v_{g_i} \to 0,$$

From this and the fact that R_{g_i} converges to R_Y strongly in L^2 over B_i we have that the right-hand side of Eq. (27) goes to zero as claimed. By elliptic regularity $\|\phi_i - 1\|_{H^2_{g_i}(B_{g_i}(p_i, \frac{2}{3}\nu_{g_i}(p_i))}$ converges to zero. As a result $(B_{g_i}(p_i, \frac{2}{3}\nu_{g_i}(p_i), p_i, g_{Y,i})$ converges weakly in H^2 to $(B_{g_\infty}(p, \frac{2}{3}\nu_{g_\infty}(p)), p, g_\infty)$. As a consequence we have that

$$\int\limits_{\Sigma} \langle \widehat{\mathrm{Ric}}_{g_{Y,i}}, h_i\rangle_{g_{Y,i}} \mathrm{d}v_{g_{Y,i}} \to \int\limits_{\Sigma_\infty} \langle \mathrm{Ric}_{g_\infty}, h\rangle_{g_\infty} \mathrm{d}v_{g_\infty},$$

for any traceless tensor h (in H^2) with support in $B_{g_\infty}(p, \nu_{g_\infty}(p)/2)$ and traceless tensors h_i (in H^2) with support in $B_{g_i}(p_i, \nu_{g_i}(p_i)/2)$ converging strongly in H^2 to h. Thus, $\delta_{h_i}\mathcal{R}|_{g=g_{Y,i}} \to \delta_h \mathcal{R}|_{g=g_\infty}$. Therefore, if $\widehat{\mathrm{Ric}}_{g_\infty} \neq 0$ in $B_{g_\infty}(p, \nu_{g_\infty}(p)/2)$

we can lower the infimum of $|Y(\Sigma)|^2$ for the functional $\mathcal{R}$ over the three-manifold Σ.

We prove now that g_∞ is complete. Let s be an incomplete geodesic in Σ_∞. Fix $p \in s$. Let S_2 be a transversal geodesic-two-simplex in Σ_∞ and having p in its interior. For $q \in s$ (in the incomplete direction and close to p) consider the three-simplex $S_3(q)$ formed by all geodesics joining q with a point in S_2. Observe that because (Ω, g_∞) is hyperbolic and s has finite length (in the incomplete direction) every $r \in \partial S_3(x)$ has a cone $C_3(r)$ inside and of size bounded below.[11] Now as q approaches the end of s, we can find a sequence of points q_i and $r(q_i) \in \partial S_3(q_i)$ with $\nu_{g_\infty}(r(q_i)) \to 0$ and having a cone inside $S_3(r(q_i))$ of size bounded below. The blow up limit of the pointed space $(\Sigma_\infty, r(q_i), \frac{1}{\nu(r(q_i))^2} g_\infty)$ has $\nu(x) = 1$ and is complete, flat and having a cone of size (α, ∞) inside. It must be $\mathbb{R}^3$ which is a contradiction.

We can conclude then that Σ_∞ consist of a finite number of connected components $H_i, i = 1, \ldots, i_H$ each one admitting a complete hyperbolic metric of finite volume $g_{H,i} = g_\infty$. Observe that it must be $\mathrm{Vol}_{g_\infty}(\Sigma_\infty) = \mathrm{Vol}_{\mathrm{inf}}$ (and not strictly less than $\mathrm{Vol}_{\mathrm{inf}}$) for otherwise (see [6]) one can find a sequence of metrics $\tilde{\tilde{g}}_i$ in Σ and with bounded $L^\infty_{\tilde{\tilde{g}}_i}$ curvature, converging to $\cup_{i=1}^{i=i_H}(H_i, g_{H,i})$ and with $\mathrm{Vol}_{\tilde{g}_i}(\Sigma) \to \mathrm{Vol}_{g_\infty}(\Sigma_\infty)$ and thus lowering the value $|Y(\Sigma)|^2$ for the infimum of $\mathcal{R}$.

Now, pick a transversal torus for each one of all the hyperbolic cusps of the Riemannian manifolds $(H_i, g_{H,i})$. Denote them by $\{T_i, i = 1, \ldots, i = i_T\}$. Each one of the tori T_i can be embedded (up to isotopy) inside Σ. As proved in [3, Theorem 2.9] if one of the tori is compressible one can again lower the infimum value for $\mathcal{R}$. Thus, the tori T_i are all incompressible. As shown in [3, p. 156] the set of tori $\{T_i, i = 1, \ldots, i_T\}$ (of a strong geometrization as this) is unique up to isotopy.

The rest of the claims in the Theorem follow from Eqs. (20)–(24). □

2.2. Examples

Examples of ground states (namely sequences $\{(g_i, K_i)\}$ of cosmologically normalized states with $\mathcal{V}_i \downarrow \mathcal{V}_{\mathrm{inf}}$ and $Q_0 \leq \Lambda$) and of the types *Case* $Y(\Sigma) = 0$ or *Case* $Y(\Sigma) < 0$ *(I)* (in Theorem 2.1) are easy to find. We will show that soon below. An example of a ground state of the type *Case* $Y(\Sigma) < 0$ *(II)* is more difficult to find and will be discussed in a separate section (Sect. 6).

Case $Y(\Sigma) = 0$. Take any two-surface Σ_{gen} of genus greater or equal than one. Consider the three-manifold $\Sigma = \Sigma_{\mathrm{gen}} \times S^1$. Denote by $l^2 \mathrm{d}s^2$ the metric on S^1 with total length l and denote by g_{gen} a metric on Σ_{gen} of scalar curvature -6. An example of a ground state of the type *Case* $Y(\Sigma) = 0$ is given by the sequence of states $\{(g_l, -g_l)\}$ on Σ where $g_l = g_{\mathrm{gen}} \times l^2 \mathrm{d}s^2$ and $l \to 0$.

[11] Given a point x in a Riemannian manifold (Σ, g) a cone of size (α, l) ($l < inj_x g$) in Σ is the image under the exponential map of a cone of size (α, l) (segments from x in $T_x\Sigma$ having length l and forming an angle α with a given segment).

Case $Y(\Sigma) < 0$ *(I)*. Take any compact hyperbolic manifold Σ with hyperbolic metric g_H. The constant sequence of states $\{(g_H, -g_H)\}$ is an example of a ground state of type *Case* $Y(\Sigma) < 0$ (*I*).

2.3. The Double Cusp

Say (H_1, g_{H_1}) and (H_2, g_{H_2}) are two complete hyperbolic metrics of finite volume and suppose that each one has, for the sake of concreteness, only one hyperbolic cusp. Denote the cusps as C_1 and C_2. Denote by $(g_{H_i}, -g_{H_i})$ the flat cone states on H_i, $i = 1, 2$. Recall that the metrics g_{H_i} on the cusps are of the form $g_{H_i} = \mathrm{d}x^2 + e^{2x} g_{T,i}$ where g_{T_i} is a flat (and x-independent) metric on the tori T^2 transversal to the cusps $(-\infty, a] \times T^2$. Consider now a torus-neck, namely the manifold $G = [-l, l] \times T^2$ with a T^2-invariant metric g_G. For any $x_0 < a$ we will find a state (g_G, K_G) on G which, at the boundary $\partial[-l, l] \times T^2 = \{-l\} \times T^2 \cup \{l\} \times T^2$, approximates to any given desired order the flat cone states of H_1 and H_2 at $x = x_0$. Once this is done we will glue $(g_{H_1}, -g_{H_1})$, (g_G, K_G) and $(g_{H_2}, -g_{H_2})$ to get an state over $H_1 \sharp G \sharp H_2$ (satisfying the constraints equations). As $x_0 \to -\infty$ these "double cusp" states display the behavior of a ground state of type *Case* $Y(\Sigma) < 0$ (*II*). A schematic picture can be seen in Fig. 1. Note that the states (g_G, K_G), being T^2-symmetric, are Gowdy and therefore explicitly tractable.

The construction is organized as follows. In Sect. 2.3.1 we find a (Gowdy) polarized space–time solution on $\mathbb{R} \times \mathbb{R} \times T^2$. Once this is done, we find in Sect. 2.3.3 a foliation of $\mathbb{R} \times \mathbb{R} \times T^2$ the states of which display (when suitable normalized) a convergence–collapse behavior of the type *Case* $Y(\Sigma) < 0$ (*II*). Although the states found in this foliation are not CMC, we will see in Sect. 2.3.5 that it is possible to find a CMC foliation the CMC states of which are not far from those found before and displaying the same convergence–collapse behavior. In Sect. 2.3.4 we find (Gowdy) non-polarized space–time solutions on $\mathbb{R} \times \mathbb{R} \times T^2$. One can then repeat the analysis done in Sects. 2.3.3 and 2.3.5 to find, for each space–time non-polarized solution, a CMC foliation displaying a convergence–collapse picture of the type *Case* $Y(\Sigma) < 0$ (*II*). The family of polarized states that we will construct is sufficient to join two arbitrary flat cone cusp sates $(C_1, (g_{H_1}, -g_{H_1}))$ and $(C_2, (g_{H_2}, -g_{H_2}))$. Suppose now we have two flat cone states $(H_i, (g_{H_i}, -g_{H_2}))$ having a hyperbolic cusp each that we want to join through a state in a torus-neck. Having fixed x_0 and a given error ϵ, suppose we have found a state (polarized or not) (g_G, K_G) in a torus-neck G, which is compatible (up to the error ϵ) at its ends with the flat cone cusps $(C_1, (g_{H_1}, -g_{H_1}))$ and $(C_2, (g_{H_2}, -g_{H_2}))$ at $x = x_0$. We will perform the gluing of $(H_1, (g_{H_1}, -g_{H_1}))$, (g_G, K_G) and $(H_2, (g_{H_2}, -g_{H_2}))$ as follows. First, we glue (keeping the T^2-symmetry) the metrics g_{H_i}, $i = 1, 2$ and g_G on an interval $([a, b] \times T^2)$ of length one in each one of the necks and centered at $x = x_0$. Denote the new metric by $g_\sharp$. Then we find a transverse traceless tensor $\hat{K}_{TT}$ with respect to $g_\sharp$ and equal to $-g_{H_i}$ or K_G outside the intervals where the metrics were glued. Using the data $(g_\sharp, \hat{K}_{TT})$ we appeal to a Theorem of Isenberg [12] to show that in the conformal class of the state $(g_\sharp, \hat{K}_{TT})$ the Lichnerowicz equation can be solved and therefore a CMC state found. Finally, we use standard elliptic estimates to show that if the error ϵ is small enough the CMC state constructed with the conformal method is as close to the states $(g_{H_i}, -g_{H_i})$ and (g_G, K_G) (in their respective domains) as we like.

2.3.1. The Geometry on a Torus Neck (the Polarized Case). On $\mathbb{R}\times\mathbb{R}\times T^2$ we look for a (polarized) T^2-symmetric space–time metric in the coordinates where it looks like

$$g = e^{2a}(-\mathrm{d}t^2 + \mathrm{d}x^2) + Re^{2W}\mathrm{d}\theta_1^2 + Re^{-2W}\mathrm{d}\theta_2^2.$$

The functions a, R, W depend on (t,x). Define the coordinates $(-,+) = (t - x, t + x)$. Derivatives with respect to $-$ and $+$ will be denoted with a subscript $+$ or $-$. In this representation the Einstein equations are equivalent to the system of scalar equations

$$\frac{\partial^2 R}{\partial x^2} - \frac{\partial^2 R}{\partial t^2} = 0, \tag{28}$$

$$\frac{\partial}{\partial t}\left(R\frac{\partial}{\partial t}W\right) - \frac{\partial}{\partial x}\left(R\frac{\partial}{\partial x}W\right) = 0, \tag{29}$$

$$2\frac{R_\pm}{R}a_\pm = \frac{R_{\pm\pm}}{R} - \frac{1}{2}\left(\frac{R_\pm}{R}\right)^2 + 2W_\pm^2. \tag{30}$$

Note that Eq. (28) is decoupled from the rest. We make the choice

$$R(x,t) = R_0(e^{2(t+x)} + e^{2(t-x)}).$$

The Eq. (29) is the Euler–Lagrange equation of the Lagrangian

$$L(t, \partial_t W, \partial_x W) = \int R(\partial_t W)^2 - R(\partial_x W)^2 \mathrm{d}x.$$

We make the choice $W(x,t) = W_1 + W_0 \arctan e^{2x}$. These solutions are the W-stable solutions, i.e. those W that with fixed values at the boundary (infinity in this case) minimize the potential $V = \int R(x,0)(\partial_x W)^2\mathrm{d}x$. We proceed now to find out a. Observing that

$$2(W_\pm)^2 = \frac{W_0^2}{2\cosh^2 2x},$$

Eq. (30) can be written

$$2\frac{R_\pm}{R}a_\pm = \frac{R_{\pm\pm}}{R} - \frac{1}{2}\left(\frac{R_\pm}{R}\right)^2 + \frac{W_0^2}{2\cosh^2 2x}. \tag{31}$$

Dividing by $R_\pm/R$ and adding and subtracting both equations we get

$$\partial_x a = -\left(\frac{1}{2} + \frac{W_0^2}{2}\right)\tanh 2x,$$

$$\partial_t a = \frac{3}{2} + \frac{W_0^2}{2},$$

which after integration give

$$a(x,t) = a(0) - \left(\frac{1}{2} + \frac{W_0^2}{2}\right)\frac{1}{2}\ln\cosh 2x + \left(\frac{3}{2} + \frac{W_0^2}{2}\right)t.$$

In the next section we analyze these solutions along some particular space-like foliations.

2.3.2. The Evolution of States on a Torus Neck. Convene that by *observes* we mean a space-like slice $\mathcal{S}(t')$ moving with a parametric time t'. Let us analyze the solutions found in the previous section with this perspective. First, for those observers who in a forced manner move keeping their x-coordinate constant and moving uniformly forward in time $t = t'$, the normalized three-geometry (normalized by $e^{\left(\frac{3}{2}+\frac{W_0^2}{2}\right)t}$), collapses along the two-tori into the one-dimensional geometry

$$g_\infty = e^{a(0)-\left(\frac{1}{2}+\frac{W_0^2}{2}\right)\frac{\ln\cosh 2x}{2}}\mathrm{d}x^2,$$

on the real line and of finite length. However, for those observers who freely fall in space–time along time-like geodesics, the normalized three-geometry will be seen to evolve into a hyperbolic cusp

$$g_\infty = \mathrm{d}x^2 + R_0 e^{2W_{\pm\infty}} e^{2x}\mathrm{d}\theta_1^2 + R_0 e^{-2W_{\pm\infty}} e^{2x}\mathrm{d}\theta_2^2.$$

There are in fact two natural sets of free-falling observers, those who move with positive x and those with negative x. Both will observe the normalized three-geometry become into hyperbolic cusps (exponentially in time). In between them the geometry is collapsing, as will be made precise in what follows.

Free falling observers. We will assume a minor approximation that in no way changes the global picture, nor the precise statements that follow on the evolution of the exact geometry. Concentrate on the region $x \geq 10$. On it the metric g (in the (t,x) plane) is almost like

$$e^{2\left(\left(\frac{3}{2}+\frac{W_0^2}{2}\right)t-\left(\frac{1}{2}+\frac{W_0^2}{2}\right)x\right)}(-\mathrm{d}t^2 + \mathrm{d}x^2).$$

We will consider time-like geodesics in this region (towards the increasing direction of t). Denote by s their proper time. Then it can be calculated that, independently of the initial velocity, the coordinates $(t(s), x(s))$ of time-like geodesics behave according to

$$-\left(\frac{1}{2} + \frac{W_0^2}{2}\right)t + \left(\frac{3}{2} + \frac{W_0^2}{2}\right)x = \frac{1}{2}\ln\frac{3+W_0^2}{1+W_0^2} + o\left(\frac{1}{s}\right),$$

$$-\left(\frac{1}{2} + \frac{W_0^2}{2}\right)x + \left(\frac{3}{2} + \frac{W_0^2}{2}\right)t = \ln s + \frac{1}{2}\ln\frac{(3+W_0^2)(1+W_0^2)}{2} + o\left(\frac{1}{s}\right).$$

What these formulas tell us is that the set of coordinates

$$t' = -\left(\frac{1}{2} + \frac{W_0^2}{2}\right)x + \left(\frac{3}{2} + \frac{W_0^2}{2}\right)t,$$
$$x' = -\left(\frac{1}{2} + \frac{W_0^2}{2}\right)t + \left(\frac{3}{2} + \frac{W_0^2}{2}\right)x,$$

form the natural coordinate system prescribed by a free-falling set of observers. In these new coordinates and after choosing $a(0) = \frac{1}{2}\ln\left(-\left(\frac{1}{2} + \frac{W_0^2}{2}\right)^2 + \left(\frac{3}{2} + \frac{W_0^2}{2}\right)^2\right)$ we get

$$g = e^{2t'}(-\mathrm{d}t'^2 + \mathrm{d}x'^2) + R_0 e^{2\left(\frac{\pi}{2}W_0 + W_1\right)}\left(e^{2(t'+x')} + e^{\frac{2}{2+W_0^2}(t'-x')}\right)\mathrm{d}\theta_1^2 + \cdots$$
$$\cdots + R_0 e^{-2\left(\frac{\pi}{2}W_0 + W_1\right)}(e^{2(t'+x')} + e^{\frac{2}{2+W_0^2}(t'-x')})\mathrm{d}\theta_2^2.$$

After making $W_{+\infty} = \frac{\pi}{2}W_0 + W_1$ and normalizing by $e^{2t'}$ we see that the local three-geometry exponentially falls into the hyperbolic cusp

$$g = \mathrm{d}x^2 + R_0 e^{2W_{+\infty}} e^{2x}\mathrm{d}\theta_1^2 + R_0 e^{-2W_{+\infty}} e^{2x}\mathrm{d}\theta_2^2$$

2.3.3. A Convergence–Collapse Picture. Let us describe now a global foliation of Cauchy hypersurfaces (labeled with a parameter $s \geq 1$) where we can see the picture of convergence–collapse. For any s the hypersurface will be defined as: (Zone I) $\{(t,x),\ -\left(\frac{1}{2} + \frac{W_0^2}{2}\right)\ln s + \left(\frac{3}{2} + \frac{W_0^2}{2}\right)t = s,\ \mid x \mid \leq \ln t\}$, (Zone II) $\{(t,x),\ s = t' = -\left(\frac{1}{2} + \frac{W_0^2}{2}\right)x + \left(\frac{3}{2} + \frac{W_0^2}{2}\right)t,\ x \geq \ln s\}$ and (Zone III) $\{(t,x),\ s = t'' = \left(\frac{1}{2} + \frac{W_0^2}{2}\right)x + \left(\frac{3}{2} + \frac{W_0^2}{2}\right)t,\ x \leq -\ln s\}$. Normalize the three-metrics over the slices with the factor e^{-2s}. As $s \to +\infty$ the limit of the normalized three-metrics are: (Zone I)

$$g_\infty = \mathrm{d}\tilde{x}^2,$$

which is the infinite-length one-dimensional geometry on the real line, and (Zone II)

$$g_\infty = \mathrm{d}x^2 + R_0 e^{2W_{+\infty}} e^{2x}\mathrm{d}\theta_1^2 + R_0 e^{-2W_{+\infty}} e^{2x}\mathrm{d}\theta_2^2,$$

on the whole $\mathbb{R} \times T^2$, and similarly for the Zone III. A schematic picture can be seen in Fig. 2.

2.3.4. The Geometry on a Torus Neck (the Non-Polarized Case). In this section we follow the same strategy as in Sect. 2.3.1 to find (Gowdy) T^2-symmetric space–time solutions but this time non-polarized. On $\mathbb{R} \times \mathbb{R} \times T^2$ we look for a non-polarized T^2-symmetric metric in the coordinates where it looks like

$$g = e^{2a}(-\mathrm{d}t^2 + \mathrm{d}x^2) + R(e^{2W} + q^2 e^{-2W})\mathrm{d}\theta_1^2 - Rqe^{-2W}2\mathrm{d}\theta_1\mathrm{d}\theta_2 + Re^{-2W}\mathrm{d}\theta_2^2, \tag{32}$$

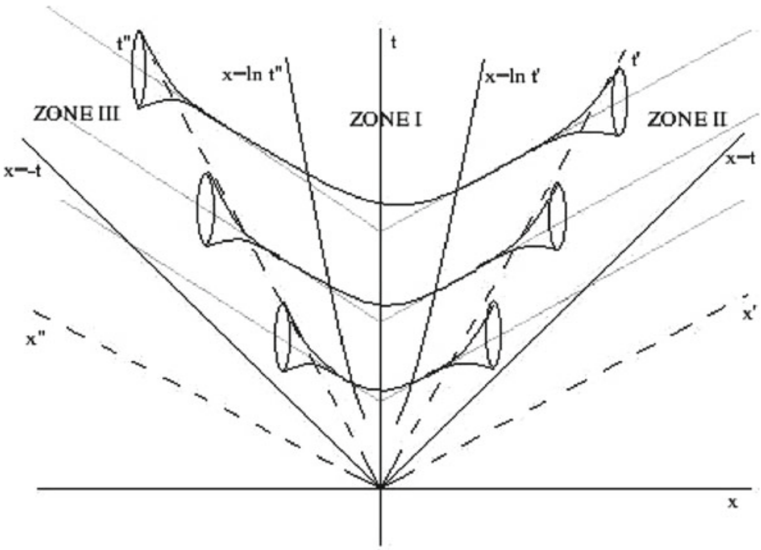

FIGURE 2. A schematic figure showing the evolution of the normalized three-geometry.

and where a, R, W depend only on (t, x) or $(u, v) = (-, +) = (t - x, t + x)$. In this representation the Einstein equations reduce to

$$R_{+-} = 0, \tag{33}$$

$$2\frac{R_{++}}{R} - \left(\frac{R_+}{R}\right)^2 + 4W_+^2 + q_+^2 e^{-4W} - 4a_+\frac{R_+}{R} = 0, \tag{34}$$

$$2\frac{R_{--}}{R} - \left(\frac{R_-}{R}\right)^2 + 4W_-^2 + q_-^2 e^{-4W} - 4a_-\frac{R_-}{R} = 0, \tag{35}$$

$$(RW_-)_+ + (RW_+)_- + Rq_+q_-e^{-4W} = 0, \tag{36}$$

$$(Re^{-4W}q_+)_- + (Re^{-4W}q_-)_+ = 0. \tag{37}$$

Again we make the choice $R(x, t) = R_0 e^{2t}\cosh(2x)$. With this choice we will solve for time-independent W and q realizing arbitrary flat metrics on the two tori at the ends, i.e. which have prescribed asymptotic q_∞, $q_{-\infty}$, W_∞, $W_{-\infty}$. After that we will solve for a.

Solving for time independent W and q. Equation (37) forces q' to satisfy

$$q' = \frac{2ce^{4W}}{\cosh(2x)}. \tag{38}$$

where c is an arbitrary constant. With q' of this form, Eq. (36) forces W to satisfy

$$W'' + 2\tanh(2x)W' = \frac{-2c^2e^{4W}}{\cosh^2(2x)}, \tag{39}$$

The strategy to find the solutions to (38)–(39) for W and q and having prescribed asymptotic values at the ends (i.e. when $x \to \pm\infty$) is the following. Fix c first. Then find W having the prescribed asymptotic values $W(\infty) = W_{\infty}$ and $W(-\infty) = W_{-\infty}$. Then vary c keeping fixed the asymptotic conditions for W and prove that we can reach at some c the prescribed asymptotic value $q(\infty) = q_{\infty}$ if $q(-\infty) = q_{-\infty}$ was prescribed. We will accomplish that by proving that varying c from some value c_0 toward zero, the integral from $-\infty$ to ∞ of Eq. (38) that defines $q(\infty)$ reaches (having $q_{-\infty}$ as the lower limit of integration prescribed) all possible values. Although Eq. (39) is highly non-linear, it can be integrated exactly. We note that Eq. (39) is equivalent (unless W is constant in which case $c = 0$ and q is constant) to

$$((\cosh(2x)W')^2)' = -(c^2 e^{4W})', \tag{40}$$

which gives

$$\cosh^2(2x)W'^2 = -c^2 e^{4W} + A^2, \tag{41}$$

for $A > 0$, an arbitrary positive constant. Taking the square root of (41) we get a separable variables ODE. After integration we get

$$W = -\frac{1}{2} \ln \frac{|\, c\,|}{A} \cosh(-2A \arctan e^{2x} + B), \tag{42}$$

with B and arbitrary constant. We need to find A and B that solve the asymptotic conditions for W i.e.

$$\frac{|\, c\,|}{A} \cosh B = e^{-2W_{-\infty}},$$

$$\frac{|\, c\,|}{A} \cosh(-\pi A + B) = e^{-2W_{\infty}}.$$

Making the change of variables $A = \frac{B-D}{\pi}$ we get the equivalent equations

$$B = D + \pi \,|\, c\,|\, e^{2W_{\infty}} \cosh D, \tag{43}$$

$$D = B - \pi \,|\, c\,|\, e^{2W_{-\infty}} \cosh B. \tag{44}$$

Now the problem is to understand the solutions B and D to (43)–(44) as functions of c, W_{∞} and $W_{-\infty}$. If we graph $B(D)$ (from 43) and $D(B)$ (from 44) on the same $B - D$-coordinates axis, we see (observe the factor $|c|$ in front of $\cosh D$ and $\cosh B$) that there is some positive c_0 above which there are no solutions (the graphs do not intersect), at which there is only one and below which there are only two solutions (see Fig. 3). In the following we will analyze the solutions A and B as $c \to 0$. We will see that given a prescribed value $q_{-\infty}$ we get any asymptotic value for q_{∞} by varying c from c_0 towards zero. The equation

$$e^{2W_{-\infty}} \cosh B = e^{2W_{\infty}} \cosh D,$$

gives for the each one of the two different branches (of solutions $(B,\, D)$) the following behaviors

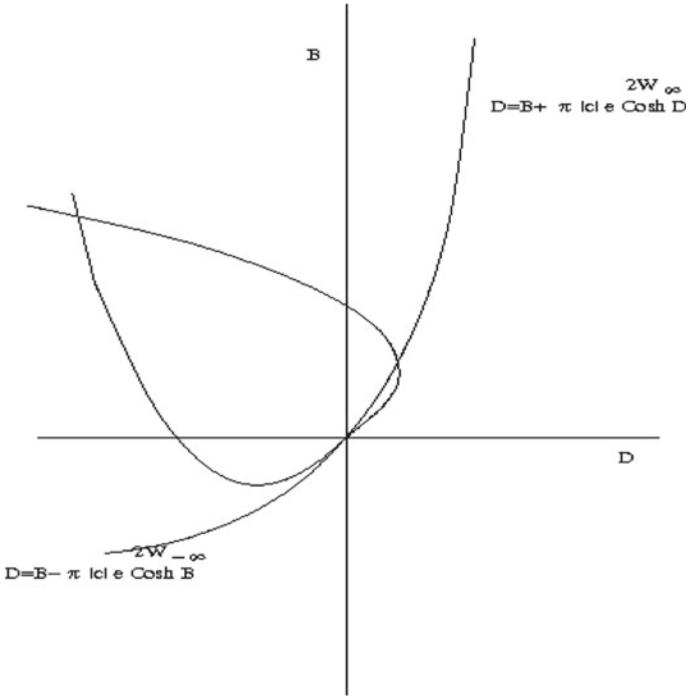

FIGURE 3. The graphs of $B(D)$ (from 43) and $D(B)$ (from 44) for a small c.

1. (Branch I). Either $W_\infty = W_{-\infty}$ for which we get (observe that $A{=}B{-}D{>}0$)

$$B = -D \to 0,$$
$$\frac{\mid c \mid}{A} \to e^{-2W_{-\infty}},$$

or $W_\infty \neq W_{-\infty}$ for which we get

$$B \to \infty \; if \; W_\infty > W_{-\infty} \; (or \; -\infty \; if \; W_\infty < W_{-\infty}),$$
$$B - D \to 2(W_\infty - W_{-\infty}) \; (or \; -2(W_\infty - W_{-\infty})),$$
$$A \to \frac{2}{\pi}(W_\infty - W_{-\infty}) \; \left(or \; -\frac{2}{\pi}(W_\infty - W_{-\infty})\right).$$

2. (Branch II) For any $W_\infty, W_{-\infty}$

$$B \to \infty, D \to -\infty,$$
$$B + D \to 2(W_\infty - W_{-\infty}),$$
$$A \sim \frac{2B - 2(W_\infty - W_{-\infty})}{\pi}.$$

With these behaviors for A and B (as $c \to 0$) we get
(Branch I). The formula for q'

$$q' = \frac{c}{(\cosh(2x))\left(\frac{|c|}{A}\cosh(-2A\arctan e^{2x} + B)\right)^2},$$

shows that, starting at an arbitrary $q_{-\infty}$, the function q approaches (uniformly) to the constant function $q = q_{-\infty}$.

(Branch II). The formula for q' approximates to

$$q' \sim \frac{\pm e^{-2W_{-\infty}}(2B - 2(W_\infty - W_{-\infty}))}{\pi\cosh B\cosh(2x)(e^{-2W_{-\infty}}(\cosh B)^{-1}\cosh(-2A\arctan e^{2x} + B))^2}.$$

Rearranged it reads

$$q' \sim \frac{\pm e^{2W_{-\infty}}(2B - 2(W_\infty - W_{-\infty}))\cosh B}{\pi\cosh(2x)\cosh(-2A\arctan e^{2x} + B))^2}. \tag{45}$$

The factor

$$\cosh(-2A\arctan e^{2x} + B) = \cosh\left(B\left(-2\frac{A}{B}\arctan e^{2x} + 1\right)\right),$$

in the denominator of Eq. (45), can be bounded above in the interval $-1 \leq x \leq 1$ by

$$\cosh 2Bx.$$

(We note that $-2\frac{A}{B} \to \frac{-4}{\pi}$, linearize $\arctan e^{2x}$ ($x \sim 0$) and get the bound). The integral

$$\pm\int_{-1}^{1}\frac{e^{-2W_{-\infty}}(2B - 2(W_\infty - W_{-\infty}))\cosh B}{\cosh 2x(\cosh 2Bx)^2}\mathrm{d}x,$$

is equal, after the change of variables $Bx = u$, to

$$\pm\int_{-B}^{B}\frac{e^{2W_{-\infty}}(2B - 2(W_\infty - W_{-\infty}))\cosh B}{B\cosh\frac{2u}{B}\cosh^2 2u}\mathrm{d}u,$$

that clearly diverges to $\pm$ infinity as B goes to infinity.

Solving for a. To find out the expression for a we follow the same procedure as in the polarized case. We find $\dot{a}$ and a' from Eqs. (34) and (35) and then integrate in time (t) and space (x). As W and q are time independent we have

$$4W_\pm^2 + q_\pm^2 e^{-W} = W'^2 + \frac{q'^2}{4}e^{-4W}.$$

Equation (41) gives

$$W'^2 + \frac{q'^2}{4}e^{-4W} = \frac{A^2}{\cosh^2 2x}.$$

This formula makes Eqs. (34) and (35) to have the same form as Eq. (31) but with W_0^2 replaced by $\frac{A^2}{2}$. This gives the following expression for a

$$a(x,t) = a(0) - \left(\frac{1}{2} + \frac{A^2}{4}\right)\frac{1}{2}\ln\cosh 2x + \left(\frac{3}{2} + \frac{A^2}{4}\right)t.$$

The analysis of the convergence–collapse picture for these non-polarized solutions follows exactly as in the polarized case.

2.3.5. The Gluing. *CMC states in a torus neck.* For simplicity we will work with the polarized solution in the torus neck we have found before the computations carry over to the non-polarized case as well. We will find a CMC slice, $t = s(x)$, of the solution

$$\begin{aligned} g &= e^{2a}(-\mathrm{d}t^2 + \mathrm{d}x^2) + Re^{2W}\mathrm{d}\theta_1^2 + Re^{-2W}\mathrm{d}\theta_2^2, \\ a(x,t) &= a(0) - \left(\frac{1}{2} + \frac{W_0^2}{2}\right)\frac{1}{2}\ln\cosh 2x + \left(\frac{3}{2} + \frac{W_0^2}{2}\right)t, \\ R(x,t) &= R_0(e^{2(t+x)} + e^{2(t-x)}), \\ W(x,t) &= W_1 + W_0 \arctan e^{2x}, \end{aligned}$$

with $k = -3$ and asymptotically of the form $t = s(x) \sim t_0 \pm \frac{(1+W_0^2)}{(3+W_0^2)}x$. With this asymptotic we guarantee having (almost) flat cone initial states on the ends. The way to find such CMC slice is by finding appropriate barriers. To do that we first find a general expression for the mean curvature of a general section $t = s(x)$. We keep the discussion brief. Given a slice $t = s(x)$ introduce a coordinate system $(\bar{x}, \bar{t}, \bar{\theta}_1, \bar{\theta}_2)$ defined as

$$\begin{aligned} x &= \bar{x} + s'(\bar{x})\bar{t}, \\ t &= s(\bar{x}) + \bar{t}, \\ \theta_1 &= \bar{\theta}_1, \\ \theta_2 &= \bar{\theta}_2. \end{aligned}$$

In these coordinates the metric g is written

$$g = -\bar{N}^2\mathrm{d}\bar{t}^2 + \bar{g}(\mathrm{d}\bar{x} + \bar{X}\mathrm{d}\bar{t})(\mathrm{d}\bar{x} + \bar{X}\mathrm{d}\bar{t}) + Re^{2W}\mathrm{d}\bar{\theta}_1^2 + Re^{-2W}\mathrm{d}\bar{\theta}_2^2,$$

where

$$\begin{aligned} \bar{g} &= e^{2a}((1 + s''\bar{t})^2 - s'^2), \\ \bar{N}^2 &= e^{2a}(1 - s'^2). \end{aligned}$$

and $\bar{X} = 0$ when $\bar{t} = 0$. From this k is calculated (at the slice $t = s(x)$) as

$$k = -\frac{1}{e^a\sqrt{1 - s'^2}}\left(\partial_{\bar{t}}a + \frac{s''}{1 - s'^2} + \frac{\partial_{\bar{t}}R}{R}\right),$$

where

$$\partial_{\bar{t}} a = \partial_t a + s' \partial_x a = -\left(\frac{1}{2} + \frac{W_0^2}{2}\right) s' \tanh 2x + \left(\frac{3}{2} + \frac{W_0^2}{2}\right),$$

$$\frac{\partial_{\bar{t}} R}{R} = \frac{\partial_x R s' + \partial_t R}{R} = 2s' \tanh 2x + 2,$$

which gives

$$k(x) = -\frac{1}{\sqrt{1-s'^2}} e^f \left(\frac{s''}{1-s'^2} - \left(\frac{1}{2} + \frac{W_0^2}{2}\right) s' \tanh 2x + \left(\frac{3}{2} + \frac{W_0^2}{2}\right) \right. \\ \left. + 2s' \tanh 2x + 2\right),$$

with

$$f = -\left(a(0) - \left(\frac{1}{2} + \frac{W_0^2}{2}\right) \frac{1}{2} \ln \cosh 2x + \left(\frac{3}{2} + \frac{W_0^2}{2}\right) s\right).$$

Remark 1. Note that $k(s(x)+\tau) = e^{-\left(\frac{3}{2}+\frac{W_0^2}{2}\right)\tau} k(s(x))$. This implies in particular that once we have obtained a CMC slice a CMC foliation is obtained by shifting it in the (t) time direction.

Now, to construct the barriers, note that for the section $t = s(x) = t_0 + \left(\frac{1+W_0^2}{3+W_0^2}\right) x$, k is asymptotically (i.e. as $x \to +\infty$) constant. A direct calculation shows that for the pair of sections (on the right end)

$$t = s_\pm(x) = t_0 + \frac{1+W_0^2}{3+W_0^2} x \pm \frac{1}{x}, \tag{46}$$

the asymptotic (to leading terms) is

$$-k \sim -k_0 e^{\mp\left(\frac{3}{2}+\frac{W_0^2}{2}\right)\frac{1}{x}} \left(1 + O\left(\frac{1}{x}\right)\right).$$

The last formula shows that $-k(s_+) < -k_0 < -k(s_-)$ asymptotically. The extension of those sections to the center of the neck can be carried as follows. Take two sections symmetric with respect to the t-axis, that (say on the right) are (i) any smooth section (s_+) from 0 to 10 with $s'' > 0$ and $s_+(10) + \frac{1+W_0^2}{3+W_0^2}(x-10) - \ln(x-9)$ thereafter (ii) any smooth section (s_-) from 0 to 10 with $s'' > 0$ and equal to $s_-(10) + \frac{1+W_0^2}{3+W_0^2}(x-10) + \ln(x-9)$ thereafter. It is easy to see using the Remark above that by shifting the section s_- upwards, at some shift the sections have disjoint range of their mean curvatures (between the points of intersection) and that at the point of intersection their tangents are $\frac{1+W_0^2}{3+W_0^2}$ up to $\sim 1/x$. Due to that, it is easy to continue these two sections as was said above (in Eq. 46), starting from an x slightly less than the x where they intersect, in such a way that they have disjoint range of mean curvatures but asymptotically approaching to $s(x) = t_0 + \frac{1+W_0^2}{3+W_0^2}$.

Note that given a CMC slice as was described above, the same slice is CMC with the same mean curvature if on the metric g we replace R_0 by $R_0 e^{-2\delta}$. Also note that on the (x', t') coordinates, for large x' the metric is written approximately

$$g = e^{2t'}(-\mathrm{d}t'^2 + \mathrm{d}x^2) + R_0 e^{2\left(\frac{\pi}{2}W_0 + W_1\right)}\left(e^{2(t'+x')} + e^{\frac{2}{2+W_0^2}(t'-x')}\right)\mathrm{d}\theta_1^2$$
$$+ R_0 e^{-2\left(\frac{\pi}{2}W_0 + W_1\right)}\left(e^{2(t'+x')} + e^{\frac{2}{2+W_0^2}(t'-x')}\right)\mathrm{d}\theta_2^2.$$

Thus, changing R_0 by $R_0 e^{-2\delta}$ and changing the x' coordinate by $x'' = x' - \delta$ the metric approximates to any given desired order to the flat cone state,

$$g = e^{2t'}(-\mathrm{d}t'^2 + \mathrm{d}x''^2) + R_0 e^{2\left(\frac{\pi}{2}W_0 + W_1\right)} e^{2(t'+x'')}\mathrm{d}\theta_1^2$$
$$+ R_0 e^{-2\left(\frac{\pi}{2}W_0 + W_1\right)} e^{2(t'+x'')}\mathrm{d}\theta_2^2.$$

Moreover, note that the distance between standard parts on the cusps get increased by $\sim 2\delta$. δ therefore parametrizes the family of CMC initial states displaying a convergence–collapse picture.

A traceless transverse tensor. Having now a metric on the torus neck we glue it to the hyperbolic metrics $dx^2 + e^{2x} g_{T_i}$ (on the right (say $i = 2$) and the left (say $i = 1$) of the neck) along intervals of left one around $x = x_0$ and preserving the T^2 symmetry. There is some freedom of course in this process. We will use it in a moment. We will look for a T^2-symmetric transverse traceless $(2, 0)$-tensor $\hat{K}_{TT}$ with respect to the metric that resulted from the gluing. Moreover, we will demand the components of $\hat{K}_{TT}$ to be zero except for $\hat{K}_{TT,xx}$, $\hat{K}_{TT,\theta_1\theta_1}$ and $\hat{K}_{TT,\theta_2\theta_2}$. Finally, we demand $\hat{K}_{TT}$ to be unchanged on the region inside the neck which is not the gluing region and, similarly, we demand $\hat{K}_{TT}$ to be unchanged inside the bulk of the hyperbolic manifolds H_1 and H_2 which is not the gluing region. Thus, we want $\hat{K}_{TT}$ to be zero on the hyperbolic sector and right after the gluing. Observing that for any T^2-symmetric metric the connection coefficients $\Gamma^{\theta_k}_{\theta_i\theta_j}$ for i, j, k equal to 0 or 1 are zero and similarly for $\Gamma^{x}_{x\theta_i}$ and $\Gamma^{\theta_i}_{xx}$ for $i = 0, 1$ we have

$$\nabla_i \hat{K}^i_{TT,\theta_j} = 0, \quad j = 0, 1.$$

For $\nabla_i \hat{K}^i_{TT\ x}$ we compute

$$\nabla_i \hat{K}^i_{TT,x} = \partial_x \hat{K}^x_{TT,x} + (\Gamma^{\theta_2}_{x\theta_2} - \Gamma^{\theta_1}_{x\theta_1})\hat{K}^x_{TT,x} \tag{47}$$

where we have implicitly used that $\hat{K}^x_{TT,x} + \hat{K}^{\theta_1}_{TT,\theta_1} + \hat{K}^{\theta_2}_{TT,\theta_2} = 0$. We need to find a solution of (47) being exactly zero after an interval of length one. To do that we choose the glued metric in such a way that $\Gamma^{\theta_1}_{x\theta_1} \neq \Gamma^{\theta_2}_{x\theta_2}$ (with a small difference) on an interval of length one half inside the gluing interval. Then choose $\hat{K}^{\theta_1\theta_1}$ such that the solution to (47) is exactly zero right after the gluing region. One can check that this can be done using the integral formula for the solution of a first order ODE.

Estimates. Once having (g, K) with $divK = 0$ and $tr_g K = k$ we invoke a theorem of Isenberg [12] guaranteeing that the Lichnerowicz equation is solvable as long as $\hat{K} \neq 0$ and $k \neq 0$ as is our case. To estimate the solution to the Lichnerowicz equation

$$\Delta\phi = \frac{1}{8}R_g\phi - \frac{1}{8}|\hat{K}|^2_g\phi^{-7} + \frac{k^2}{12}\phi^5,$$

we use the maximum principle and the standard local elliptic estimates. From the maximum principle we get

$$R_g\phi(x_{\max}) - |\hat{K}|^2\phi(x_{\max})^{-7} + \frac{k^2}{12}\phi(x_{\max})^5 \leq 0,$$

Now note that $R_g = |\hat{K}|^2 - \frac{2}{3}k^2 + \epsilon(x)$ where $\epsilon(x)$ is nonzero only on the gluing region. Using this in the last equation gives

$$|\hat{K}|^2(\phi(x_{\max}) - \phi^{-7}(x_{\max})) + \frac{2}{3}k^2(\phi(x_{\max})^5 - \phi(x_{\max})) + \epsilon(x_{\max})\phi(x_{max}) \leq 0. \quad (48)$$

Observe that $\|K\|_{L^\infty_g}$ is bounded with a bound independent of ϵ. We see from Eq. (48) that when $\|\epsilon\|_{L^\infty} \to 0$ then $\|\phi - 1\|_{L^\infty} \to 0$. Standard elliptic estimates show that in fact $\|\phi - 1\|_{C^{2,\alpha}} \to 0$.

3. Long-Time Geometrization of the Einstein Flow

3.1. The Long-Time Geometrization of the Einstein Flow

In this section we prove the following Theorem.

Theorem 4. *Let Σ be a compact three-manifold with $Y(\Sigma) \leq 0$. Say $(\tilde{g}, \tilde{K})(\sigma)$ is a cosmologically normalized flow with $\tilde{\mathcal{E}}_1(\sigma) \leq \Lambda$ where Λ is a positive constant. Then, the cosmologically normalized flow $(\tilde{g}, \tilde{K})(\sigma)$ persistently geometrizes the manifold Σ. Moreover the induced geometrization is the Thurston geometrization iff $\mathcal{V}(\sigma) \downarrow \mathcal{V}_{inf} = \left(-\frac{1}{6}Y(\Sigma)\right)^{\frac{3}{2}}$.*

We need some preliminary propositions.

Proposition 2. *Let Σ be a compact three-manifold. Say g_0 is a H^2-Riemannian-metric on Σ. Say $p \in \Sigma$ and $2R < r_2(p)$ where $r_2(p)$ is the H^2-harmonic radius of the metric g_0 at the point p. According to the definition of H^2-harmonic radius we consider a harmonic coordinate system $\{x\}$ covering $B_{g_0}(p, r_2(p))$ and satisfying*

$$\frac{3}{4}\delta_{jk} \leq g_{0,jk} \leq \frac{4}{3}\delta_{jk}, \quad (49)$$

$$r_2(p)\left(\sum_{|I|=2,j,k}\int_{B_{g_0}(p,r_2(p))}\left|\frac{\partial^I}{\partial x^I}g_{jk}\right|^2 \mathrm{d}v_x\right) \leq 1. \quad (50)$$

Then there is $\epsilon(R)$ *such that if* $\|g-g_0\|_{H^2_{\{x\}}(B_{g_0}(p,R))} \leq \bar{\epsilon} \leq \epsilon(R)$ *the inclusions* $id : H^i_g(B_{g_0}(p,R)) \hookrightarrow H^i_{g_0}(B_{g_0}(p,R))$ *and* $id : H^i_{g_0}(B_{g_0}(p,R)) \hookrightarrow H^i_g(B_{g_0}(p,R))$ *for* $i=0,1,2$ *have norms controlled by* $\bar{\epsilon}$ *and* R.

Proof. Note first the Sobolev embeddings[12]

$$H^1_{\{x\}}(B_{g_0}(p,R)) \hookrightarrow L^4_{\{x\}}(B_{g_0}(p,R)), \tag{51}$$

$$H^2_{\{x\}}(B_{g_0}(p,R)) \hookrightarrow C^0_{\{x\}}(B_{g_0}(p,R)). \tag{52}$$

From (51) we see that $\|g-g_0\|_{C^0_{\{x\}}(B_{g_0}(p,R))} \leq \epsilon'(\bar{\epsilon},R)$ with $\epsilon' \to 0$ as $\bar{\epsilon} \to 0$ (and R fixed). This in particular implies that

$$C_1 g_{0,ij} \leq g_{ij} \leq C_2 g_{0,ij},$$

where C_1 and C_2 depend on $\bar{\epsilon}$ and R and tend to one as $\bar{\epsilon} \to 0$ (keeping R fixed). This proves the inequality

$$C_1\|U\|_{L^2_{g_0}(B_{g_0}(p,R))} \leq \|U\|_{L^2_g(B_{g_0}(p,R))} \leq C_2\|U\|_{L^2_{g_0}(B_{g_0}(p,R)},$$

for some C_1 and C_2 dependent on $\bar{\epsilon}$ and R, which terminates the case $i=0$. In the following we will use the notation C_1, C_2 to denote generic quantities depending on $\bar{\epsilon}$ and R. Let us prove the case $i=1$ now. Denote by ∇ and $\bar{\nabla}$ the covariant derivatives associated to g_0 and g respectively. Write $\bar{\nabla} = \nabla + \Gamma$. With this notation we have

$$|\bar{\nabla}U|^2_g = |\nabla U + \Gamma * U|^2_g \leq C_2(|\nabla U|^2_{g_0} + |\Gamma|^2_{g_0}|U|^2_{g_0}).$$

Integrating we get

$$\int\limits_{B_{g_0}(p,R)} |\bar{\nabla}U|^2_g \mathrm{d}v_g$$

$$\leq C_2\left(\int\limits_{B_{g_0}(p,R)} |\nabla U|^2_{g_0}\mathrm{d}v_{g_0} + \left(\int\limits_{B_{g_0}(p,R)} |\Gamma|^4_{\{x\}}\mathrm{d}v_x\right)^{\frac{1}{2}}\left(\int\limits_{B_{g_0}(p,R)} |U|^4_{g_0}\mathrm{d}v_{g_0}\right)^{\frac{1}{2}}\right). \tag{53}$$

It is direct to see from the formula

$$\Gamma^k_{ij} = \frac{1}{2}(\nabla_i(g_{jm}-g_{0,jm}) + \nabla_j(g_{im}-g_{0,im}) - \nabla_m(g_{ij}-g_{0,ij}))g^{km},$$

that $\|\Gamma\|_{H^1_{\{x\}}(B_{g_0}(p,R))} \to 0$ as $\bar{\epsilon} \to 0$. Sobolev embeddings applied to equation (53) give

$$\|\bar{\nabla}U\|^2_{L^2_g(B_{g_0}(p,R))} \leq C_2\|U\|^2_{H^1_{g_0}(B_{g_0}(p,R))},$$

[12] It is crucial that the embeddings are from $H^\star_{\{x\}}(B_{g_0}(p,R))$ and not from $H^\star_{0,\{x\}}(B_{g_0}(p,R))$. This is justified by the fact that, in the coordinate system $\{x\}$ the set $B_{g_0}(p,R)$ has the cone property at its boundary (see [9, p. 158]).

and thus

$$\|U\|^2_{H^1_g(B_{g_0}(p,R))} \leq C_2 \|U\|^2_{H^1_{g_0}(B_{g_0}(p,R))},$$

as desired. Let us prove the other inequality. Write

$$|\nabla U|^2_{g_0} = |\bar{\nabla} U - \Gamma * U|^2_{g_0} \leq C_1(|\bar{\nabla} U|^2_g + |\Gamma|^2_{g_0}|U|^2_{g_0}).$$

Integrating we get

$$\int_{B_{g_0}(p,R)} |\nabla U|^2_{g_0} \mathrm{d}v_{g_0}$$

$$\leq C_1 \left(\int_{B_{g_0}(p,R)} |\bar{\nabla} U|^2_g \mathrm{d}v_g + \left(\int_{B_{g_0}(p,R)} |\Gamma|^4_{g_0} dv_{g_0} \right)^{\frac{1}{2}} \left(\int_{B_{g_0}(p,R)} |U|^4_{g_0} \right)^{\frac{1}{2}} \right),$$

Again Sobolev embeddings give

$$\|\nabla U\|^2_{L^2_{g_0}(B_{g_0}(p,R))} \leq C_1(\|\bar{\nabla} U\|^2_{L^2_g(B_{g_0}(p,R))} + \|\Gamma\|^2_{H^1_{\{x\}}(B_{g_0}(p,R))} \|U\|^2_{H^1_{g_0}(B_{g_0}(p,R))}).$$

Moving the second term on the right hand side to the left side and choosing $\bar{\epsilon}$ sufficiently small[13] we have

$$\|U\|_{H^1_{g_0}(B_{g_0}(p,R))} \leq C_1 \|U\|_{H^1_g(B_{g_0}(p,R))},$$

as desired. The case $i = 2$ follows easily from the case $i = 1$. □

We consider now the Einstein flow with zero shift, i.e. we assume we have set $X = 0$.

Proposition 3. (Continuity of the flow) *Say Σ is a compact three-manifold with $Y(\Sigma) \leq 0$. Say $(g, K)(k)$ is a long-time Einstein flow with domain (at least) $[-3, 0)$. Suppose that $\mathcal{E}_1(k) \leq \Lambda$ where Λ is a positive constant. We use the notation $(g_0, K_0) = (g(-3), K(-3))$, $k_0 = -3$ and $\mathcal{V}(-3) = \mathcal{V}_0$. Say $p \in \Sigma$ and $r_{2,g_0}(p) \geq 2R$. Then for any $\epsilon > 0$ there is $\delta k(\Lambda, \mathcal{V}_0, R) > 0$ such that*

$$\sup_{k \in [k_0, k_0 + \delta k]} \{\|(g, K)(k) - (g, K)(k_0)\|_{H^2_{g_0}(B_{g_0}(p,R)) \times H^1_{g_0}(B_{g_0}(p,R))}\} \leq \epsilon.$$

Remark 2. i. Proposition 3 would be self evident if we have a priori control on r_{2,g_0} over the whole manifold Σ. It is not a priori clear how is that the regions where the harmonic radius (or volume radius) is small may affect the evolution of the regions where it is not, even in the short time. What Proposition 3 shows is that under an a priori bound in $\mathcal{E}_1$ this influence is not noticeable in a definite interval of time $t = k$ (depending on $\mathcal{E}_1$, ν_0 and R). Note however that we do not make any claim about the continuity in $H^2_{g_0}(B_{g_0}(p, R))$ of the lapse N. As we will remark later the $H^2_{\tilde{g}_0}(B_{g_0}(p, R))$ norm of N is indeed controlled but we do not know whether N satisfies a continuity of the type claimed for g and K (in their respective spaces). In particular we do not have

[13] Note that C_1 does not blow up as $\bar{\epsilon} \to 0$.

any estimation (in any norm) of the time derivative of N on $B_{g_0}(p, R)$ even for short times. This issue will appear later in Proposition 4.

ii. The Proposition 3 is evidently true if we use the cosmologically normalized variables $(\tilde{g}, \tilde{K})$, σ and $\tilde{\mathcal{E}_1}$ instead of the variables (g, K), k and $\mathcal{E}_1$.

Proof. The crucial fact is to note that there are $\delta(R, \|\mathrm{Ric}\|_{L^2_g(\Sigma)})$ and $\epsilon(R, \|\mathrm{Ric}\|_{L^2_g(\Sigma)})$ such that if $\|g(k) - g_0\|_{H^2_{g_0}(B_{g_0}(p,R))} \leq \epsilon$ then $r_{2,g_k}(\partial B_{g_0}(p, R)) \geq \delta$. This result can be easily proved by contradiction or simply invoking the discussion in [1, see p. 218, p. 227]. Recall that

$$\|\mathrm{Ric}\|^2_{L^2_g(\Sigma)} \leq C(|k|\mathcal{V} + Q_0),$$

where C is a numeric constant. As a result the H^2-harmonic radius of the region $B_{g(k)}(B_{g_0}(p, R), \frac{2}{3}\delta)$ is controlled from below by $\Lambda, \mathcal{V}_0$ and R as long as $\|g(k) - g_0\|_{H^2_{g_0}(B_{g_0}(p,R))} \leq \epsilon$. Elliptic regularity shows that the norms $\|\hat{K}\|_{H^2_{g(k)}(B_{g_0}(p,R))}$, $\|N\|_{H^3_{g(k)}(B_{g_0}(p,R))}$, $\|E_0\|_{H^1_{g(k)}(B_{g_0}(p,R))}$ and $\|B_1\|_{H^1_{g(k)}(B_{g_0}(p,R))}$ are controlled from above by Λ, $\mathcal{V}_0$ and R as long as $\|g(k) - g_0\|_{H^2_{g_0}(B_{g_0}(p,R))} \leq \epsilon$. Under zero shift, the time derivatives of g and K are

$$\begin{aligned} g^{\cdot} &= -2NK, \\ K^{\cdot} &= -\nabla^2 N + N(E - K \circ K). \end{aligned}$$

Thus $\|g^{\cdot}\|_{H^2_{g(k)}(B_{g_0}(p,R))}$ and $\|K^{\cdot}\|_{H^1_{g_0}(B_{g_0}(p,R))}$ are controlled above by (say) $\tilde{\Lambda}(\Lambda, \mathcal{V}_0, R)$ as long as $\|g(k) - g_0\|_{H^2_{g_0}(B_{g_0}(p,R))} \leq \epsilon$. Write

$$\|g(k) - g_0\|_{H^2_{g_0}(B_{g_0}(p,R))} \leq \int_{k_0}^{k} \|g^{\cdot}\|_{H^2_{g_0}(B_{g_0}(p,R))} \mathrm{d}k,$$

$$\|K(k) - K_0\|_{H^1_{g_0}(B_{g_0}(p,R))} \leq \int_{k_0}^{k} \|K^{\cdot}\|_{H^1_{g_0}(B_{g_0}(p,R))} \mathrm{d}k.$$

By Proposition 2 we can bound $\|g^{\cdot}\|_{H^2_{g_0}(B_{g_0}(p,R))}$ by $C_1 \|g^{\cdot}\|_{H^2_{g(k)}(B_{g_0}(p,R))}$ and similarly for the $H^1_{g_0}$-norm of $K^{\cdot}$. Thus the length δk of the maximal interval $[k_0, k_0 + \delta k]$ where $\|g(k) - g_0\|_{H^2_{g_0}(B_{g_0}(p,R))} \leq \epsilon$ is greater than $\epsilon/(C_1 \tilde{\Lambda})$ and similarly for the $H^1_{g_0}$-norm of K. □

Proposition 4. *Let Σ be a compact three-manifold with $Y(\Sigma) \leq 0$. Assume $(\tilde{g}, \tilde{K})$ is a cosmologically normalized long-time flow. Assume too that $\tilde{\mathcal{E}_1} \leq \Lambda$ with Λ a positive constant. Then, for every $\epsilon > 0$ and $R > 0$ there exists σ_0 such that for* *any* $\sigma \geq \sigma_0$ *and* $p \in \Sigma$ *with* $r_{2,\tilde{g}(\sigma)} \geq 4R$ *we have*

$$\|\hat{\tilde{K}}(\sigma)\|_{H^1_{\tilde{g}(\sigma)}(B_{\tilde{g}(\sigma)}(p,R))} \leq \epsilon, \tag{54}$$

$$\|\widehat{\mathrm{Ric}}(\sigma)\|_{L^2_{\tilde{g}(\sigma)}(B_{\tilde{g}(\sigma)}(p,R))} \leq \epsilon, \tag{55}$$

$$\|E_0(\sigma)\|^2_{L^2_{\tilde{g}(\sigma)}(B_{\tilde{g}(\sigma)}(p,R))} + \|B_0(\sigma)\|^2_{L^2_{\tilde{g}(\sigma)}(B_{\tilde{g}(\sigma)}(p,R))} \leq \epsilon. \tag{56}$$

Proof. The way to prove Proposition 4 is to show that for any $R > 0$, any sequence of points $\{p_i\}$, and any divergent sequences of logarithmic-times $\{\sigma_i\}$ for which $r_{2,\tilde{g}(\sigma_i)} \geq 4R$, the norms (54), (55) and (56) (with σ_i instead of σ and p_i instead of p) tend to zero. We will use the terminology "*Case* 54" for the proof of this on $\hat{\tilde{K}}$ and similarly for $\widehat{\mathrm{Ric}}$ (Case 55) and $E_0,\ B_0$ (Case 56).

Let us start by making some elementary but important observations.

Observation 1. From (the proof of) Proposition 3 we know that there are $\{\delta\sigma_i\}$ with $|\delta\sigma_i|$ controlled from below by Λ, $\mathcal{V}$ and R (observe that because $\mathcal{V}$ is monotonic along the flow we can replace the dependence on $\mathcal{V}(\sigma_i)$ for the dependence only on $\mathcal{V}_0 = \mathcal{V}(\sigma_0)$ with σ_0 some initial logarithmic time) and such that the norms $\|\hat{\tilde{K}}\|_{H^2_{\tilde{g}(\sigma)}(B_{\tilde{g}(\sigma_i)}(p_i,2R))}$ for $\sigma \in [\sigma_i, \sigma_i + \delta\sigma_i]$, are controlled from above by Λ, $\mathcal{V}_0$ and R. It follows from the maximum principle applied to the lapse equation

$$-\Delta_{\tilde{g}(\sigma)}\tilde{N} + |\tilde{K}(\sigma)|^2_{\tilde{g}(\sigma)}\tilde{N} = 1,$$

that $\tilde{N}(p,\sigma) \geq \tilde{N}_0(\Lambda, \mathcal{V}_0, R) > 0$ for p in $B_{\tilde{g}(\sigma_i)}(p_i, \frac{7}{4}R)$ and for σ in $[\sigma_i, \sigma_i + \delta\sigma_i]$.[14]

Observation 2. Recall that

$$\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\sigma} = 3\int\limits_{\Sigma} 3\tilde{N} - 1\mathrm{d}v_{\tilde{g}} = 3\int\limits_{\Sigma} \phi \mathrm{d}v_{\tilde{g}},$$

where (as was introduced in the background) $\phi = 3\tilde{N} - 1$ is the Newtonian potential and satisfies $-1 \leq \phi \leq 0$. If we integrate this equation between σ_i and $\sigma_i + \delta\sigma_i$ (where $\delta\sigma_i$ will be the one in Proposition 3) we get

$$\begin{aligned}\mathcal{V}(\sigma_i) - \mathcal{V}(\sigma_i + \delta\sigma_i) &= -3\int\limits_{\sigma_i}^{\sigma_i+\delta\sigma_i}\int\limits_{\Sigma} \phi(\sigma)\mathrm{d}v_{\tilde{g}(\sigma)}\mathrm{d}\sigma\\ &\geq -3\int\limits_{\sigma_i}^{\sigma_i+\delta\sigma_i}\int\limits_{B_{\tilde{g}(\sigma_i)}(p_i,2R)} \phi(\sigma)\mathrm{d}v_{\tilde{g}(\sigma)}\mathrm{d}\sigma.\end{aligned}$$

[14] The argument is by contradiction. Assume there exists a sequence of states violating the inequality an obtain a convergent sub-sequence which violated the maximum principle.

As $\mathcal{V}(\Sigma_i) - \mathcal{V}(\sigma_i + \delta\sigma_i) \to 0$ when $\sigma_i \to \infty$ (because $\mathcal{V}$ is monotonic and greater than zero) it follows that

$$\mu\{\sigma \in [\sigma_i, \sigma_i + \delta\sigma_i] / \left(\int_{B_{\tilde{g}(\sigma_i)}(p_i,2R)} \phi^2(\sigma) \mathrm{d}v_{\tilde{g}(\sigma)} \right) > \Gamma\} \to 0,$$

as $\sigma_i \to \infty$, and for any fixed $\Gamma > 0$.

Let us prove now that $\|\hat{\tilde{K}}\|_{L^2_{\tilde{g}(\sigma_i)}(B_{\tilde{g}(\sigma_i)}(p_i,\frac{7}{4}R))} \to 0$ as $\sigma_i \to \infty$. Recall that

$$\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\sigma} = -3 \int_{\Sigma} \tilde{N} |\hat{\tilde{K}}|^2_{\tilde{g}} \mathrm{d}v_{\tilde{g}}.$$

Integrating in σ we have

$$\mathcal{V}(\sigma_i) - \mathcal{V}(\sigma_i + \delta\sigma_i) \geq 3 \int_{\sigma_i}^{\sigma_i+\delta\sigma_i} \int_{B_{\tilde{g}(\sigma_i)}(p_i,\frac{7}{4}R)} \tilde{N} |\hat{\tilde{K}}|^2_{\tilde{g}} \mathrm{d}v_{\tilde{g}}.$$

It follows from Proposition 3 and *Observation 1* that $\mathcal{V}(\sigma)$ can get below its limit $\mathcal{V}_\infty = \lim_{\sigma\to\infty} \mathcal{V}(\sigma)$ unless $\lim_{\sigma\to\infty} \|\hat{\tilde{K}}\|_{L^2_{\tilde{g}(\sigma_i)}(B_{\tilde{g}(\sigma_i)}(p_i,\frac{7}{4}R))} = 0$. Now as $\|\hat{\tilde{K}}\|_{H^2_{\tilde{g}(\sigma_i)}(B_{\tilde{g}(\sigma_i)}(p_i,2R))}$ is controlled from above by Λ, $\mathcal{V}$ and R, it follows that if $\|\hat{\tilde{K}}\|_{H^1_{\tilde{g}(\sigma_i)}(B_{\tilde{g}(\sigma_i)}(p_i,\frac{7}{4}R))} \geq M > 0$ we can extract a sub-sequence of the pointed spaces $(B_{\tilde{g}(\sigma_i)}(p_i, \frac{7}{4}R), p_i, \tilde{g}(\sigma_i))$ converging to a limit space (strongly in H^2) $(B_{\tilde{g}_\infty}(p_\infty, \frac{7}{4}R), p_\infty, \tilde{g}_\infty)$ where $\hat{\tilde{K}}$ is not converging to zero which is a contradiction. This finishes the case (54).

We use now this result and *Observation 1* to get an improved version of *Observation 1*.

Observation 3. Local elliptic estimates applied to the lapse equation (in the ϕ-variable)

$$\Delta_{\tilde{g}}\phi - |\tilde{K}|^2_{\tilde{g}}\phi = |\hat{\tilde{K}}|^2_{\tilde{g}},$$

give

$$\mu\{\sigma \in [\sigma_i, \sigma_i + \delta\sigma_i] / \|\phi\|_{H^2_{\tilde{g}(\sigma)}(B_{\tilde{g}(\sigma_i)}(p_i,\frac{3}{2}R))}(\sigma) \geq \Gamma\} \to 0,$$

as $\sigma_i \to \infty$ and for any fixed $\Gamma > 0$. An important consequence of this is that for any space-like tensors U_k, $k = 1, 2, 3$ such that $\|U_k\|_{L^2_{\tilde{g}(\sigma)}(B_{\tilde{g}(\sigma_i)}(p_i,\frac{3}{2}R))} \leq M$ for some $M > 0$ and for any $k = 1, 2, 3$ we have

$$\left| \int_{\sigma_i}^{\sigma_i+\delta\sigma_i} \int_{B_{\tilde{g}(\sigma_i)}(p_i,\frac{3}{2}R)} U_0 * \phi + U_1 * \nabla\phi + U_3 * \nabla^2\phi \; dv_{\tilde{g}(\sigma)} d\sigma \right| \to 0,$$

as $\sigma_i \to \infty$.

Recalling that

$$\mathrm{Curl}_{\tilde{g}} \tilde{\hat{K}} = -B_0,$$

we conclude that $\|B_0\|_{L^2_{\tilde{g}(\sigma_i)}(B_{\tilde{g}(\sigma_i)}(p_i,2R))} \to 0$ (which is "half" the case (56)).

To prove the case (55) we note that it is enough from

$$\widehat{\mathrm{Ric}}_{\tilde{g}} = E + \hat{\tilde{K}} + \hat{\tilde{K}} \circ \hat{\tilde{K}} - \frac{1}{3}|\hat{\tilde{K}}|^2 \tilde{g},$$

and case (54), to prove that $\|E\|_{L^2_{\tilde{g}(\sigma_i)}(B_{\tilde{g}(\sigma_i)}(p_i,R))}$ tends to zero as $\sigma_i \to \infty$. This is however more difficult than the cases before. We will study the quantity

$$\int_{B_{\tilde{g}(\sigma_i)}\left(p_i,\frac{3}{2}R\right)} \langle E, \hat{\tilde{K}} \rangle_{\tilde{g}} \mathrm{d}v_g,$$

and its time derivative with respect to logarithmic time. Differentiating with respect to σ we have

$$\left(\int_{B_{\tilde{g}(\sigma_i)}(p_i,\frac{3}{2}R)} \langle E, \hat{\tilde{K}} \rangle_{\tilde{g}} dv_{\tilde{g}} \right)^{\cdot} = \int_{B_{\tilde{g}(\sigma_i)}\left(p_i,\frac{3}{2}R\right)} \langle E; \hat{\tilde{K}} \rangle_{\tilde{g}} + \langle E, \hat{\tilde{K}} \rangle_{\tilde{g}}$$
$$- \langle E \circ \hat{\tilde{K}}, \tilde{g}^{\cdot} \rangle_{\tilde{g}} + 3 \langle E, \hat{\tilde{K}} \rangle_{\tilde{g}} \phi \mathrm{d}v_{\tilde{g}}. \tag{57}$$

To get a more convenient expression of the right hand side of the previous equation we will use the following expressions for the time derivatives of the cosmologically normalized variables $\tilde{g}$, E and $\hat{\tilde{K}}$

$$\dot{\tilde{g}} = 2\phi \tilde{g} - 6 \tilde{N} \hat{\tilde{K}}, \tag{58}$$

$$\dot{E} = \tilde{N} \mathrm{Curl}_{\tilde{g}} B - \frac{\nabla \tilde{N}}{\tilde{N}} \wedge_{\tilde{g}} B - \frac{5}{2} E \times_{\tilde{g}} \tilde{K} - \frac{2}{3} \langle E, \tilde{K} \rangle_{\tilde{g}} \tilde{g} - \frac{3}{2} E, \tag{59}$$

$$\dot{\hat{\tilde{K}}} = -\hat{\tilde{K}} - \phi \tilde{g} - \nabla^2 \phi + \phi E + E - \tilde{N} (\hat{\tilde{K}} \circ \hat{\tilde{K}} - 2 \hat{\tilde{K}}). \tag{60}$$

We now integrate Eq. (57) in σ for σ in $[\sigma_i, \sigma_i]$. After integration of the left hand side we have (naturally) the expression

$$\left(\int_{B_{\tilde{g}(\sigma_i)}\left(p_i,\frac{3}{2}R\right)} \langle E, \hat{\tilde{K}} \rangle_{\tilde{g}} \mathrm{d}v_{\tilde{g}} \right) (\sigma_i + \delta\sigma_i) - \left(\int_{B_{\tilde{g}(\sigma_i)}\left(p_i,\frac{3}{2}R\right)} \langle E, \hat{\tilde{K}} \rangle_{\tilde{g}} \mathrm{d}v_{\tilde{g}} \right) (\sigma_i). \tag{61}$$

From Case (54) and the bound

$$\left| \int_{B_{\tilde{g}(\sigma_i)}\left(p_i,\frac{3}{2}R\right)} \langle E, \hat{\tilde{K}} \rangle_{\tilde{g}} \mathrm{d}v_{\tilde{g}} \right| (\sigma) \leq \|E_0\|_{L^2_{\tilde{g}}\left(B_{\tilde{g}(\sigma_i)}\left(p_i,\frac{3}{2}R\right)\right)}(\sigma) \|\hat{\tilde{K}}\|_{L^2_{\tilde{g}}\left(B_{\tilde{g}(\sigma_i)}\left(p_i,\frac{3}{2}R\right)\right)}(\sigma),$$

we get that for any σ (in particular for $\sigma = \sigma_i$ and $\sigma = \sigma_i + \delta\sigma_i$) we have that (61) tends to zero as $i \to \infty$. Similarly, using either *Observation 3*, Case (54) or the B_0-part of Case (56) we have that all the terms in the right hand side of the integral in σ of Eq. (57), except perhaps the term

$$\int_{\sigma_i}^{\sigma_i+\delta\sigma_i} \int_{B_{\tilde{g}(\sigma_i)}(p_i,\frac{3}{2}R)} |E_0|^2 dv_{\tilde{g}} \mathrm{d}\sigma,$$

tend to zero. Thus we are lead to conclude that this term also tends to zero when $i \to \infty$. We will see now using the Gauss equation that $(\int_{B_{\tilde{g}(\sigma_i)}(p_i,R)} |E_0|^2 \mathrm{d}v_{\tilde{g}})(\sigma_i)$ tends to zero as $i \to \infty$. That would finish Case (56) and Case (55). The argument is as follows. Consider a fixed, even and positive function f of one variable x, equal to zero for $|x| \geq \frac{3}{2}$ and equal to one for $|x| \leq 1$. Consider the function $f(r)$ where r is the geodesic radius from p_i and corresponding to the metric $\tilde{g}(\sigma_i)$ inside $B_{\tilde{g}(\sigma_i)}(p_i, 2R)$. Extend $f(r)$ to the space–time in such a way that it is time independent. Consider finally the Weyl field $\mathbf{W} = f\mathbf{Rm}$. We have

$$E_W = fE_0,\ B_W = fB_0,$$

and

$$\mathbf{J}_{\mathbf{W},bcd} = (\boldsymbol{\nabla}^a f)\mathbf{Rm}_{abcd}.$$

Thus, the $L^2_{\tilde{g}}(B_{\tilde{g}(\sigma_i)}(p_i, \frac{3}{2}R))$ norm of $E_{\mathbf{W}}$, $B_{\mathbf{W}}$ and $\mathbf{J}_{\mathbf{W}}$ are controlled by Λ, $\mathcal{V}_0$ and R. It follows from integrating the Gauss equation

$$\tilde{Q}(\mathbf{W})^{\cdot} = \tilde{Q}(\mathbf{W}) - 9\int_{\Sigma} \tilde{N}\tilde{Q}(\mathbf{W})_{\alpha\beta\tilde{T}\tilde{T}}\tilde{\mathbf{\Pi}}^{\alpha\beta}\mathrm{d}v_{\tilde{g}}.$$

in σ and from σ_i to $\sigma_i + \delta\sigma$ that

$$|\tilde{Q}(\mathbf{W})(\sigma_i + \delta\sigma) - \tilde{Q}(\mathbf{W})(\sigma_i)| \leq \tilde{\Lambda}(\Lambda, \mathcal{V}_0, R)\delta\sigma.$$

Thus if $(\int_{B_{\tilde{g}(\sigma_i)}(p_i,R)} |E_0|^2_{\tilde{g}}\mathrm{d}v_{\tilde{g}})(\sigma_i) \geq M > 0$ we can choose $\delta\sigma$ such that for all σ in $[\sigma_i, \sigma_i + \delta\sigma]$ it is $\tilde{Q}(\mathbf{W})(\sigma) \geq \frac{M}{2} > 0$. But we have

$$\tilde{Q}(\mathbf{W}) = \int_{B_{\tilde{g}(\sigma_i)}(p_i,\frac{3}{2}R)} f^2(|E_0|^2 + |B_0|^2)\mathrm{d}v_{\tilde{g}},$$

and we know from the B_0-part of Case (56) that

$$\lim_{\sigma_i\to\infty} \sup_{\sigma\in[\sigma_i,\sigma_i+\delta\sigma]} \left\{ \left(\int_{B_{\tilde{g}(\sigma_i)}(p_i,\frac{3}{2}R)} f^2|B_0|^2\mathrm{d}v_{\tilde{g}} \right)(\sigma) \right\} \to 0,$$

when $i \to \infty$. Therefore, if σ_i is big enough

$$\left(\int_{B_{\tilde{g}(\sigma_i)}(p_i, \frac{3}{2}R)} |E_0|^2 \mathrm{d}v_{\tilde{g}} \right)(\sigma) \geq \frac{M}{3}$$

for any σ in $[\sigma_i, \sigma_i + \delta\sigma]$ which would contradict that

$$\int_{\sigma_i}^{\sigma_i+\delta\sigma_i} \int_{B_{\tilde{g}(\sigma_i)}(p_i, \frac{3}{2}R)} |E_0|^2 \mathrm{d}v_{\tilde{g}} d\sigma,$$

tends to zero as σ_i tends to infinity. □

We are ready to prove Theorem 4. The proof goes essentially along the same lines as the proof of the geometrization of the flow given in [15] for long-time flows under $C^{\alpha}_{\tilde{g}}$ curvature bounds. We repeat it here for the sake of clarity.

Proof (*of Theorem 4*). We prove first there is a divergence sequence of logarithmic times $\{\sigma_i\}$ with $\left(\Sigma^{\frac{1}{i}}, (\tilde{g}, \tilde{K})(\sigma_i)\right)$ converging to $\cup_{i=1}^{i=n}(H_i, (\tilde{g}_{H,i}, -\tilde{g}_{H,i}))$ (weakly in H^2). Introduce a new variable $j = 1, 2, 3, \ldots$. For $j = 1$ find a sequence $\{\sigma_{1,i}\}$ with $(\Sigma^1, \tilde{g}(\sigma_{1,i}))$ convergent weakly in H^2. For $j = 2$ find a sub-sequence $\{\sigma_{2,i}\}$ of $\{\sigma_{1,i}\}$ with $(\Sigma^{1/2}, \tilde{g}(\sigma_{2,i}))$ convergent in the weak H^2 topology. Proceed similarly for all j to have a double sequence $\{\sigma_{j,i}\}$. Now, for the diagonal sequence $\{\sigma_{i,i}\}$, $(\Sigma^{1/i}, \tilde{g}(\sigma_{i,i}))$ converges into a union of Riemannian manifolds of finite volume, denoted as $\cup_\nu(M_\nu, \tilde{g}_{\infty,\nu})$. By Proposition 4, $\tilde{K}(\sigma_{i,i})$ converges strongly to $-\tilde{g}_{\infty,\nu}$ in H^1. Also by Proposition 4 we get that each metric $\tilde{g}_{\infty,\nu}$ is hyperbolic and the convergence is in the strong H^2-topology. Therefore, as there is a lower bound for the volume of complete hyperbolic manifolds of finite volume and the total volume of the limit space is bounded above, there must be a finite number of components, and we can write $\cup_\nu(M_\nu, \tilde{g}_{\infty,\nu}) = \cup_{i=1}^{i=n}(H_i, \tilde{g}_{H,i})$.

We prove next that each component $(H_j, \tilde{g}_{H,j})$ is persistent. For simplicity assume there is only one component and therefore $(\Sigma^{1/i}, \tilde{g}(\sigma_{i,i}))$ converges in the strong H^2-topology to $(H, \tilde{g}_H)$. There are two possibilities according to whether the component is compact or not, we discuss them separately.

1.(*The compact case*) Assume $(H, \tilde{g}_H)$ is compact. Consider the space of metrics $\mathcal{M}_H$ in H. For every metric g consider the orbit of g under the diffeomorphism group (of H^3-diffeomorphisms). Denote such orbit by $o(g)$. Around $\tilde{g}_H$ consider a small (smooth) section $\mathcal{S}$ of $\mathcal{M}_H$ (made of $H^2_{\tilde{g}_H}$ metrics) and transversal to the orbits generated by the action on $\mathcal{M}_H$ of the diffeomorphism group.[15] If ϵ_0 is sufficiently small every metric g in $\mathcal{M}_H$ with $\|g - \tilde{g}_H\|_{H^2_{\tilde{g}_H}} \leq \epsilon_0$ can be uniquely projected into $\mathcal{S}$ by a diffeomorphism, or in other words we can consider the projection

[15] Which particular section is taken is unimportant. One can use for instance $\mathcal{S} = \{g/id : (H, g) \to (H, \tilde{g}_H)\}$ is harmonic (see [5,11]).

$P(g) = o(g) \cap \mathcal{S}$. Note that one can project every flow of metrics $\tilde{g}(t)$ starting close to $\tilde{g}_H$, to a path $P(\tilde{g}(t))$, until at least the first time when $\|P(\tilde{g}(t)) - \tilde{g}_H\|_{H^2_{\tilde{g}_H}} = \epsilon_0$ or in other words until at least when the projection touches the boundary of the ball of center $\tilde{g}_H$ and radius ϵ_0 in $H^2_{\tilde{g}_H}$ (denote such ball as $B(\tilde{g}_H, \epsilon_0)$).

Recall Mostow rigidity.[16]

Mostow rigidity (*the compact case*). There is ϵ_1 such that if $P(g'_H) \in B(\tilde{g}_H, \epsilon_1)$, where g'_H is a hyperbolic metric in H then $P(g'_H) = \tilde{g}_H$.

Fix $\epsilon_2 = min\{\epsilon_0, \epsilon_1\}$. Observe that as $\tilde{g}_{\sigma_{i,i}} \to \tilde{g}_H$ in H^2 there is a sequence of diffeomorphisms ϕ_i such that $\phi_i^*(g(\sigma_{i,i}))$ converges to $\tilde{g}_H$ in $H^2_{\tilde{g}_H}$. Now, if the geometrization is not persistent there is $\epsilon \leq \epsilon_2$ and i_2 such that if $i \geq i_2$ then $P(\phi_i^*(\tilde{g}(\sigma)))$ is well defined for $\sigma \geq \sigma_{i,i}$ until a first time $\sigma_{i,i} + T_i$ when $P(\phi_i^*(\tilde{g}(\sigma_{i,i} + T_i)))$ is in $\partial B(\tilde{g}_H, \epsilon_2)$. But we know the sequence of Riemannian manifolds $(H, P(\phi_i^*(\tilde{g}(\sigma_{i,i} + T_i))))$ converge in H^2 to $\tilde{g}_H$, and that means by the definition of H^2 convergence and Mostow rigidity that there is a sequence of diffeomorphisms φ_i such that $P(\varphi_i^*(P(\phi_i^*(\tilde{g}(\sigma_{i,i} + T_i)))))$ converges to $\tilde{g}_H$ in $H^2_{\tilde{g}_H}$. This contradict the fact that $P(\phi_i^*(\tilde{g}(\sigma_{i,i} + T_i)))$ is in $\partial B(\tilde{g}_H, \epsilon_2)$.

2. (*The non-compact case*). The proof of this case proceeds along the same lines as in the compact case but special care must be taken at the cusps.[17] Let us assume for simplicity that there is only one cusp in the piece $(H, \tilde{g}_H)$. Given A sufficiently small there is a unique torus transversal to the cusp, to be denoted by T^2_A, of constant mean curvature and area A. Denote by H_A the "bulk" side of the torus T^2_A in H. Consider the set $\mathcal{M}_{H_A}$ of metrics $\tilde{g}$ on H_A such that $\tilde{g} = \tilde{g}_H$ on $B_{\tilde{g}_H}(T^2_A, 1)$. Consider the action of the diffeomorphism group (of H^3-diffeomorphisms) on $\mathcal{M}_{H_A}$ and leaving $B_{\tilde{g}_H}(T^2_A, 1)$ invariant. Again the orbit of a metric $\tilde{g}$ will be denoted by $o(\tilde{g})$. Consider a small (smooth) section $\mathcal{S}$ of $\mathcal{M}_{H_A}$ transversal to the orbits of the action by the diffeomorphism group mentioned above. Finally consider the projection $P(\tilde{g}) = o(\tilde{g}) \cap \mathcal{S}$ which is well defined on a ball $B(\tilde{g}_H, \epsilon_0)$ for ϵ_0 small enough. Observe again that a flow of metrics $\tilde{g}(t)$ in $\mathcal{M}_{H_A}$ can be projected into $\mathcal{S}$ until at least the first time when $P(\tilde{g}(t))$ is in $\partial B_{\tilde{g}_H}(\tilde{g}_H, \epsilon_0)$. Slightly abusing the notation (as we would require a pointed sequence) consider the sequence $(\Sigma, \tilde{g}(\sigma_{i,i}))$ converging in H^2 to $\tilde{g}_H$. There is a sequence of diffeomorphisms (onto the image) $\phi_{\sigma_{i,i}} : H_A \to \Sigma$ such that $\|\phi^*_{\sigma_{i,i}}(\tilde{g}(\sigma_{i,i})) - \tilde{g}_H\|_{H^2_{\tilde{g}_H}}$ converges to zero. Note that if we have a map $\phi_\sigma : H_A \to \Sigma$ such that $\|\phi^*_\sigma \tilde{g}(\sigma) - \tilde{g}_H\|_{H^2_{\tilde{g}_H}} \leq 2\epsilon$ for ϵ sufficiently small then we can deform $\phi^*_\sigma \tilde{g}(\sigma)$ to a metric $S(\phi^*_\sigma \tilde{g}(\sigma))$ in $\mathcal{M}_{H_A}$ in

[16] Mostow rigidity states that any two hyperbolic metrics on a compact manifold are necessarily isometric. What we state as *Mostow rigidity* here is an obvious consequence of this fact. For the notions on hyperbolic three-geometry that we will need we refer the reader to the beautiful survey by Gromov [10]. Most of the treatment of hyperbolic three-geometry we will perform here goes in parallel to a similar analysis in the Ricci flow in [11].

[17] In [15] we have used CMC tori (transversal to the cusp) of a given area to compare (in a unique way) the Riemannian spaces $(H, \tilde{g}_H)$ and $(\Sigma, \tilde{g}(\sigma))$ (see [15] for details). If $\tilde{g}(\sigma)$ is close to $\tilde{g}_H$ only in $H^2_{\tilde{g}_H}$ the CMC tori of a given area and transversal to the tori may be difficult to guarantee. It is for this reason that (see later in the text) we smooth out the metrics $\tilde{g}(\sigma)$ near the regions where "the CMC tori of a given area would be".

such a way that (a) $S(\phi_\sigma^*\tilde{g}(\sigma)) = \phi_\sigma^*\tilde{g}(\sigma)$ on H_{e^4A}, (b) inside $(H_{e^2A} - H_{e^4A})$ the metric $S(\phi_\sigma^*\tilde{g}(\sigma))$ is chosen to minimize the $L^2_{S(\phi_\sigma^*\tilde{g}(\sigma))}$-norm of the traceless part of its Ricci tensor. Note the following elementary fact: if ϵ is chosen small enough and we have a diffeomorphism $\phi : (H_A, \tilde{g}_H) \to (\Sigma, \tilde{g})$ with $\|\phi_\sigma^*\tilde{g} - \tilde{g}_H\|_{H^2_{\tilde{g}_H}} \leq 2\epsilon$ and $\phi^*\tilde{g}$ is isometric to $\tilde{g}_H$ then the new metric $S(\phi^*\tilde{g})$ is the deformation of $\tilde{g}_H$ by a diffeomorphism on H_A (we will recall this note later as *note N*).

We note now the following crucial facts (justified below).

i. For all $A > 0$ but sufficiently small there exists σ_0 and ϵ_0 such that for all $\epsilon \leq \epsilon_0$ and $\sigma_1 \geq \sigma_0$ if there exists $\phi_{\sigma_1} : H_{e^4A} \to \Sigma$ with $\|\phi_{\sigma_1}^*\tilde{g}(\sigma_1) - \tilde{g}_H\|_{H^2_{\tilde{g}_H}} \leq \epsilon$ then there exists $\bar{\phi}_{\sigma_1} : H_A \to \Sigma$ with $\|\bar{\phi}_{\sigma_1}^*\tilde{g}(\sigma_1) - \tilde{g}_H\|_{H^2_{\tilde{g}_H}} \leq 2\epsilon$. Note that in this case $S(\bar{\phi}_\sigma^*\tilde{g}(\sigma))$ is well defined.

ii. From i. we conclude that if we have $\phi_{\sigma_1} : H_A \to \Sigma$ with satisfying: (a) $\|\phi_{\sigma_1}^*\tilde{g}(\sigma_1) - \tilde{g}_H\|_{H^2_{\tilde{g}_H}} \leq 2\epsilon$, (b) the restriction of ϕ_{σ_1} to H_{e^4A} with $\|\phi_{\sigma_1}^*\tilde{g}(\sigma_1) - \tilde{g}_H\|_{H^2_{\tilde{g}_H}(H_{e^4A})} \leq \epsilon$, (c) $\|P(S(\phi_{\sigma_1}^*\tilde{g}(\sigma_1))) - \tilde{g}_H\|_{H^2_{\tilde{g}}} \leq \epsilon$ then $\phi_\sigma : H_A \to \Sigma$ with the properties (a) and (b) exist for $\sigma \geq \sigma_1$ (and varying continuously) until at least the first time σ_2 for which $\|P(S(\phi_{\sigma_2}^*\tilde{g}(\sigma_2))) - g_H\|_{H^2_{\tilde{g}_H}} = \epsilon$.

Let us justify now claim i. Recall Mostow rigidity.

Mostow rigidity (*the non-compact case*). There is A_0 such that for any $A \leq A_0$ there is ϵ_0 such that if (Σ', g'_H) is a complete hyperbolic manifold of finite volume and $\phi : H_A \to \Sigma'$ is a diffeomorphism onto the image satisfying $\|\phi^*(g'_H) - \tilde{g}_H\|_{H^2_{\tilde{g}_H}} \leq \epsilon_0$ then (Σ', g'_H) is isometric to $(H, \tilde{g}_H)$.[18]

The justification of i. follows straight from Mostow rigidity. Indeed pick any A such that $e^2A \leq A_0$ and $\epsilon \leq \epsilon_0$ as in the Mostow rigidity statement. Suppose there exists a divergent sequence $\{\sigma_i\}$ and a sequence of diffeomorphisms onto the image $\phi_{\sigma_i} : H_{e^4A} \to \Sigma$ such that $\|\phi_{\sigma_i}^*\tilde{g}(\sigma_i) - \tilde{g}_H\|_{H^2_{\tilde{g}_H}} \leq \epsilon$ but such that it cannot be extended to a diffeomorphism $\bar{\phi}_{\sigma_i} : H_A \to \Sigma$ with $\|\bar{\phi}_{\sigma_i}^* - \tilde{g}_H\|_{H^2_{\tilde{g}_H}} \leq 2\epsilon$. We can extract a (pointed) sub-sequence of $\{(\Sigma, \tilde{g}(\sigma_i)\}$ converging to a complete hyperbolic metric of finite volume, which, by Mostow rigidity and the choice of A and ϵ must be isometric to $\tilde{g}_H$. Therefore for σ_i sufficiently big the diffeomorphism $\bar{\phi}_{\sigma_i}$ can be defined which is a contradiction.

Now from facts i. and ii. we get that, if the geometrization $(H, \tilde{g}_H)$ is not persistent there is $\epsilon \leq \epsilon_0$ and σ_0 such that if $\sigma_{i,i} \geq \sigma_0$ then $P(S\phi_\sigma^*(\tilde{g}(\sigma)))$ is well defined for $\sigma \geq \sigma_{i,i}$ until a first time $\sigma_{i,i} + T_i$ when $P(S(\phi_i^*\tilde{g}(\sigma_{i,i} + T_i)))$ is in $\partial B(\tilde{g}_H, \epsilon)$. Now the sequence $\phi_i^*(\tilde{g}(\sigma_{i,i} + T_i))$ has a sub-sequence converging in H^2 to a complete hyperbolic metric of finite volume. Again as in the compact case,

[18] The justification of this claim is as follows. According to the Mostow-Prasad rigidity g' and $\tilde{g}_H$ will be isometric if we can prove that Σ' is diffeomorphic to H. If ϵ is chosen small enough this is equivalent to show that the number of cusps of Σ and Σ' are the same. This follows from the Margulis lemma [10] and the fact that if $\|\phi^*g' - \tilde{g}_H\|_{H^2_{\tilde{g}_H}} \leq \epsilon$ then $\|\phi^*g' - \tilde{g}_H\|_{C^{\frac{1}{2}}_{\tilde{g}_H}} \leq C\epsilon$ where C is a numeric constant.

by Mostow rigidity it must be converging in H^2 to $\tilde{g}_H$. Therefore (recall *note N*) $P(S(\phi^*_\sigma \tilde{g}(\sigma)))$ must be converging to a metric on H_A which is a diffeomorphism of $\tilde{g}_H$ contradicting the fact that $P(S(\phi^*_i(\tilde{g}(\sigma_{i,i}+T_i))))$ is in $\partial B_{\tilde{g}_H}(\tilde{g}_H,\epsilon_2)$.

To finish the proof of the persistence of the geometrization one still needs to show that the compliment of the persistent pieces $(H_i,\tilde{g}_{H,i})$ is the G sector or in other words that for any $\epsilon>0$, $(\Sigma^\epsilon(\sigma),\tilde{g}(\sigma))$ converges to the ϵ-thick part of the persistent pieces $(H_i,\tilde{g}_{H,i})$. The proof of this fact follows by contradiction. If this is not the case one can extract a divergent sequence of logarithmic times containing an H-piece different from the pieces $(H_i,\tilde{g}_{H,i})$. One can prove again that this new piece is persistent leading into a contradiction for if persistent, the piece must be one of the pieces $(H_i,\tilde{g}_{H,i})$ by the way these pieces are defined. □

3.2. Stability of the flat cone (Case $Y(\Sigma)<0$ (I)-ground state)

In this section we will prove the stability of the *Case* $Y(\Sigma)<0$ *(I)*-ground state. Namely, we will show that a cosmologically normalized flow $(\tilde{g},\tilde{K})$ over a hyperbolic three-manifold Σ, with initial data $(\tilde{g},\tilde{K})(\sigma_0)$ close (in $H^3\times H^2$) to the ground state $(g_H,-g_H)$, converges (in $H^3\times H^2$) to the ground state $(g_H,-g_H)$ when $\sigma\rightarrow\infty$. This stability has been proved by Andersson and Moncrief in [5] (for rigid hyperbolic manifolds Σ).[19]

Theorem 5. (Stability of the flat cone). *Let Σ be a compact hyperbolic three-manifold. Then, there is an $\epsilon>0$ such that the cosmologically normalized CMC flow $(\tilde{g},\tilde{K})(\sigma)$ of a cosmologically normalized $(H^3\times H^2)$ initial state $(g_0,K_0)=(\tilde{g}(\sigma_0),\tilde{K}(\sigma_0))$ with $\tilde{\mathcal{E}}_1(\sigma_0)+(\mathcal{V}-\mathcal{V}_{inf})\leq\epsilon$, converges in $H^3_{g_H}\times H^2_{g_H}$ (and for a suitable choice of the shift vector X) to $(g_H,-g_H)$ (the standard Case $Y(\Sigma)<0$ (I)-ground state).*

Remark 3. As it is stated Theorem 5 gives few information about the shift vector X. This inconvenient can be remedied if, as in [5], X is chosen in such a way that for every σ the identity $id:(\Sigma,\tilde{g}(\sigma))\rightarrow(\Sigma,g_H)$ is a harmonic map (the *spatially harmonic gauge* [5]). Full control of the evolution of the shift vector X can be obtained in this case.

We begin with a preliminary Proposition.

Proposition 5. *Say Σ is a compact hyperbolic three-manifold. Fix $\underline{\nu}_0>0$ and $\mathcal{V}_0>\mathcal{V}_{inf}$. Then, for every $\epsilon>0$ there is $\delta(\epsilon,\underline{\nu}_0,\mathcal{V}_0)>0$ such that for every cosmologically normalized state $(\tilde{g},\tilde{K})$ with $\underline{\nu}\geq\underline{\nu}_0$, $\mathcal{V}\leq\mathcal{V}_0$ and $\|\hat{K}\|_{L^2_{\tilde{g}}}+\tilde{Q}_0\leq\delta$ we have $\mathcal{V}-\mathcal{V}_{inf}\leq\epsilon$.*

Proof. It is enough to prove that any sequence $(\tilde{g},\tilde{K})$ (we will forget about putting sub-index) with $\|\hat{\tilde{K}}\|_{L^2_{\tilde{g}}}+\tilde{Q}_0\rightarrow 0$ has a sub-sequence converging in H^2 to g_H.

[19] The rigidity condition is a somehow mild restriction. We remove it with an appropriate use of the reduced volume. The core of the proof is essentially the same as in [5].

From Proposition 8 in the Appendix we have (for arbitrary states (g, K))

$$\left(\int_M 2|\nabla \hat{K}|^2 + |\hat{K}|^4 \mathrm{d}v_g\right)^{\frac{1}{2}} \leq C\left(|k| \|\hat{K}\|_{L^2_g} + Q_0^{\frac{1}{2}}\right).$$

Thus $\|\hat{\tilde{K}}\|_{L^4_{\tilde{g}}} \to 0$ as $\|\hat{\tilde{K}}\|_{L^2_{\tilde{g}}} + \tilde{Q}_0 \to 0$. From this and

$$\widehat{\mathrm{Ric}}_{\tilde{g}} = E + \hat{\tilde{K}} + \hat{\tilde{K}} \circ \hat{\tilde{K}} - \frac{1}{3}|\hat{\tilde{K}}|^2 \tilde{g},$$

we get that $\|\widehat{\mathrm{Ric}}_{\tilde{g}}\|_{L^2_{\tilde{g}}} \to 0$. Moreover from the energy constraint we get $\|R_{\tilde{g}} + 6\|_{L^2_{\tilde{g}}} \to 0$. As $\mathcal{V}$ is bounded above and $\underline{\nu}$ bounded below, there is a subsequence of $(\tilde{g}, \tilde{K})$ converging (in H^2) to g_H. Thus $\mathcal{V} \to \mathcal{V}_{inf}$. $\square$

Proof (*of theorem 5*). Recall from Theorem 3 (and elliptic estimates) that for any $\epsilon > 0$ there is $\delta > 0$ such that if for a cosmologically normalized state $(\tilde{g}, \tilde{K})$ it is $\tilde{\mathcal{E}}_1 + (\mathcal{V} - \mathcal{V}_{inf}) \leq \delta$ then there is a diffeomorphism ϕ such that $(\phi^*(\tilde{g}), \phi^*(\tilde{K}))$ is ϵ-close to $(g_H, -g_H)$ in $H^3_{g_H} \times H^2_{g_H}$. One can also find $\delta > 0$ such that in addition the $L^\infty_{\tilde{g}}$-norm of the deformation tensor $\mathbf{\Pi}_{ab} = \mathbf{\nabla}_a T_b$ (with respect to the CMC foliation) is less than ϵ. It is direct to see [5] that this implies the following inequality for the evolution of $\tilde{\mathcal{E}}_1$

$$\partial_\sigma \tilde{\mathcal{E}}_1 \leq -\left(2 - C\tilde{\mathcal{E}}_1^{\frac{1}{2}}\right)\tilde{\mathcal{E}}_1. \tag{62}$$

Thus, this inequality and the monotonicity of the reduced volume show that, as long as the flow is defined it will remain close in $H^3 \times H^2$ to the ground state[20] (and thus the volume radius is controlled). As there is a lower bound for the time interval for which flows are defined if the initial data is close in $H^3 \times H^2$ to the ground state[21] we conclude that the flow is a long-time flow. Note that the argument is independent of the shift X. One may well take the zero shift $X = 0$. Now, it is clear from Eq. (62), that $\tilde{\mathcal{E}}_1 \to 0$ as the logarithmic time diverges. To show that (up to diffeomorphism) the flow $(\tilde{g}, \tilde{K})$ converges (in $H^3_{g_H} \times H^2_{g_H}$) to $(g_H, -g_H)$ it remains to prove that $\mathcal{V} - \mathcal{V}_{inf} \to 0$. By Proposition 5 if $\mathcal{V}(\sigma) - \mathcal{V}_{inf} \geq \Gamma > 0$ for all σ (observe that $\mathcal{V}$ is monotonically decreasing) it must be $\|\hat{\tilde{K}}\|_{L^2_{\tilde{g}}}(\sigma) \geq M > 0$ (for some $M > 0$) for all $\sigma \geq \sigma_1$. If ϵ is chosen small enough it must be $\|\tilde{N}(\sigma) - \frac{1}{3}\|_{L^\infty} < \frac{1}{6}$ for all $\sigma \geq \sigma_0$. The equation for the evolution of the reduced volume

$$\frac{d\mathcal{V}}{d\sigma} = -3\int_\Sigma \tilde{N}|\hat{\tilde{K}}|^2 \mathrm{d}v_{\tilde{g}}, \tag{63}$$

[20] Note again, as was explained at the end of the Introduction, that here "close in $H^3 \times H^2$" means close up to diffeomorphism.

[21] See Theorem 1 in [16] for more details and also [5] for a continuity criteria in the harmonic gauge.

shows that if $\|\tilde{\hat{K}}\|_{L^2_{\tilde{g}}}(\sigma) \geq M > 0$ for $\sigma \geq \sigma_1$ then $\mathcal{V} - \mathcal{V}_{\text{inf}}$ must go below zero after some time which is a contradiction. □

4. Hyperbolic Rigidity, Ground States and Gravitational Waves

There are several theoretical reasons to believe that the reduced volume $\mathcal{V}$ should decrease to its infimum $\mathcal{V}_{\text{inf}} = \left(-\frac{1}{6}Y(\Sigma)\right)^{\frac{3}{2}}$ at least for solutions in the family of long time solutions having a uniform bound on $\tilde{\mathcal{E}}_1$. It may be possible (see [15]) to prove this claim for long time solutions having uniform bounds on the $C^{\alpha}_{\tilde{g}}$-norm of (the electric and magnetic parts of the) space–time Riemann tensor. Proving the claim for solutions in the family of long-time solutions having a uniform bound in $\tilde{\mathcal{E}}_1$ could be a task of much greater difficulty. In this section we present various facts and arguments pointing to the validity of this claim.

According to Margulis, hyperbolic cusps are rigid in the following sense: if a complete hyperbolic metric $\tilde{g}_H$ on a manifold $\mathbb{R} \times T^2$ is close enough to a hyperbolic cusp metric $\tilde{g}_C = dx^2 + e^{2x} g_{T^2}$ over a domain $\Omega = [-a, \infty) \times T^2$ with a positive and big enough, then $\tilde{g}_H$ is isometric to $\tilde{g}_C$.

Consider the following spaces

$$DC = \{\tilde{g} \text{ on } \mathbb{R}\times T^2/R_{\tilde{g}} \geq -6,\ \tilde{g} \sim \tilde{g}_R \text{ when } x \to \infty \text{ and} \tilde{g} \sim \tilde{g}_L \text{ when } x \to -\infty\},$$
$$SC = \{\tilde{g} \in \text{on } D \times S^1/R_{\tilde{g}} \geq -6,\ \text{and } g \sim \tilde{g}_{S,R} \text{ when } x \to \infty\},$$

where DC accounts for "double cusps" and SC for "single cusp". $\tilde{g}_R$ and $\tilde{g}_L$ are two arbitrary but fixed hyperbolic cusp metrics on the (right and left) ends of $\mathbb{R} \times T^2$ and $\tilde{g}_{S,R}$ is an arbitrary but fixed metric on the (right) end of $D \times S^1$ (D is the unit two-dimensional disc). Consider now two cosmologically normalized flow $(\tilde{g}_{DC}, \tilde{K}_{DC})$ and $(\tilde{g}_{SC}, \tilde{K}_{SC})$ over $\mathbb{R} \times T^2$ and $D \times S^1$ respectively and with $\tilde{g}_{DC}$ in DC and $\tilde{g}_{SC}$ in SC (see Fig. 4). As the states evolve one may argue that they lose "energy" (actually they lose reduced volume) by the emission of cylindrical gravitational waves[22] at the ends of the cusps. According to Margulis the states would settle into the infinite double cusp (for the flow $(\tilde{g}_{DC}, \tilde{K}_{DC})$) or the infinite single cusp (for the flow $(\tilde{g}_{SC}, \tilde{K}_{SC})$) if it were the case that these configurations are $\mathcal{V}$-rigid. This is indeed true for the double cusp (a ground state) but false for the single cusp (a non-ground state) in the following sense.

Proposition 6. *Consider the set of metrics $\tilde{g}$ in DC with $\tilde{g} \sim \tilde{g}_R$[23] for $x \in [a_R, \infty)$ and $\tilde{g} \sim \tilde{g}_L$ for $x \in (-\infty, a_L]$. Call $\mathcal{V}_R$ the volume of $\tilde{g}_R$ on the region $(-\infty, a_R] \times T^2$ and similarly for the left cusp ($\mathcal{V}_L$). Then the volume of $\tilde{g}$ on the region $[a_L, a_R] \times T^2$ is strictly greater than $\mathcal{V}_L + \mathcal{V}_R$.*

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[22] In the definition of the sets DC and SC we can assume the metrics $\tilde{g}$ are T^2-symmetric. That would justify the statement that the system emits cylindrical gravitational waves.

[23] A precise meaning for $\sim$ can be given.

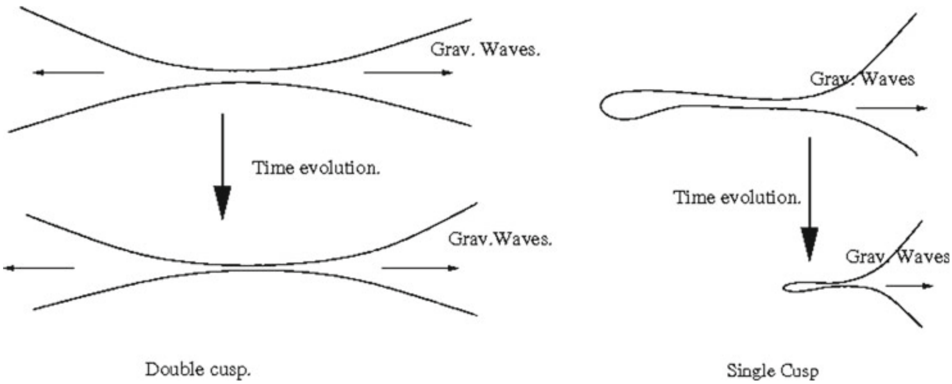

Figure 4. The (conjectural) evolution of the Double Cusp and Single Cusp.

Proposition 7. *Consider the set of metrics $\tilde{g}$ in SC with $\tilde{g} = \tilde{g}_{S,R}$ for $x \in [a_R, \infty)$. Call $\mathcal{V}_R$ the volume of $\tilde{g}_{S,R}$ on the region $(-\infty, a_R] \times T^2$. Then there exist metrics $\tilde{g}$ as described above and having volume inside the region $(-\infty, a_R] \times T^2$ less than $\mathcal{V}_R$.*

A proof of Proposition 7 and an explicit construction of such metrics is given in [13] (the metrics are indeed T^2-symmetric). It can be seen analytically (and numerically) that as time evolves the evolution of the (Yamabe) initial states $(\tilde{g}_0, -\tilde{g}_0)$ described in [13] actually separates from the single infinite hyperbolic cusp (as it should be).

Acknowledgments

I would like to thank Michael Anderson for his constant enthusiasm and support in the present work and also for suggesting the problem of the Double-Cusp ground state and its evolution.

Appendix

We begin by recalling a useful formula from [7]. Let V a symmetric traceless $(2,0)$ tensor with

$$(\text{div } V)_a = \nabla_b V^b_{\ a} = \rho,$$
$$(\text{curl } V)_{ab} = \frac{1}{2}(\epsilon_a^{\ lm}\nabla_l V_{mb} + \epsilon_b^{\ lm}\nabla_l V_{ma}) = \sigma,$$

then

$$\int_\Sigma |\nabla V|^2 + 3\langle \text{Ric}, V \circ V\rangle - \frac{1}{2}R|V|^2 = \int_\Sigma |\sigma|^2 + \frac{1}{2}|\rho|^2. \tag{64}$$

Proposition 8. *Say* Σ *is a compact three-manifold. Then* Q_0 *and* $|k|^2\|\hat{K}\|^2_{L^2_g}$ *control* $\|\nabla\hat{K}\|^2_{L^2_g}$, $\|\hat{K}\|^4_{L^4_g}$ *and* $\|\widehat{\mathrm{Ric}}\|^2_{L^2}$. *More in particular we have*

$$\left(\int_M 2|\nabla\hat{K}|^2+|\hat{K}|^4 \mathrm{d}v_g\right)^{\frac{1}{2}} \leq C\left(|k|\|\hat{K}\|_{L^2_g}+Q_0^{\frac{1}{2}}\right), \qquad (65)$$

where C *is a numeric constant.*

Observe the absence of the volume in Eq. (65) and that all norms involved are intrinsic.

Proof. Substituting $\mathrm{Ric}=E-kK+K\circ K$, $K=\hat{K}+\frac{k}{3}g$ and $V=\hat{K}$ in Eq. (64) we get

$$\int_\Sigma |\nabla\hat{K}|^2+\frac{5}{2}|\hat{K}|^4-k\langle\hat{K},\hat{K}\circ\hat{K}\rangle-\frac{k^2}{3}|\hat{K}|^2+3\langle E,\hat{K}\circ\hat{K}\rangle \mathrm{d}v_g=\int_\Sigma |B|^2\mathrm{d}v_g.$$

This equation gives the bound

$$\int_\Sigma |\nabla\hat{K}|^2+|\hat{K}|^4\mathrm{d}v_g \leq C\int_\Sigma (|k|^2|\hat{K}|^2+|k||\hat{K}|^3+|\hat{K}|^2|E|+|B|^2)\mathrm{d}v_g, \qquad (66)$$

Observe now that the inequalities

$$\int_\Sigma |\hat{K}|^2(|E|^2+|B|^2)^{\frac{1}{2}}\mathrm{d}v_g \leq \left(\int_\Sigma |\hat{K}|^4\mathrm{d}v_g\right)^{\frac{1}{2}} Q_0^{\frac{1}{2}},$$

$$\int_\Sigma |\hat{K}|^3\mathrm{d}v_g \leq \left(\int_\Sigma |\hat{K}|^2\mathrm{d}v_g\right)^{\frac{1}{2}}\left(\int_\Sigma |\hat{K}|^4\mathrm{d}v_g\right)^{\frac{1}{2}},$$

transform Eq. (66) into

$$2\|\nabla\hat{K}\|^2_{L^2_g}+\|\hat{K}\|^4_{L^4_g}-C\left(|k|\|\hat{K}\|_{L^2_g}+Q_0^{\frac{1}{2}}\right)\|\hat{K}\|^2_{L^4_g}-C(|k|^2\|\hat{K}\|^2_{L^2_g}+Q_0)\leq 0.$$

Now make $x^2=2\int_\Sigma |\nabla\hat{K}|^2+|\hat{K}|^4\mathrm{d}v_g$, $a=\left(|k|\|\hat{K}\|_{L^2_g}+Q_0^{\frac{1}{2}}\right)$ in the last equation. We get

$$x^2-Cax-Ca^2\leq 0.$$

Solving for x in the inequality above we get Eq. (65) which finishes the proof. □

The next proposition relates $\|\hat{K}\|_{L^2_g}$ with $\mathcal{V}-\mathcal{V}_{inf}$ or $\mathcal{V}$ depending on the signature of the Yamabe invariant $Y(\Sigma)$.

Proposition 9. *Say Σ is a compact three-manifold. Then*

i) if $Y(\Sigma) > 0$, $|k|^2 \int_\Sigma |\hat{K}|^2 \mathrm{d}v_g \leq C|k|^{\frac{1}{2}} \mathcal{V}^{\frac{1}{2}} (\int_\Sigma |\hat{K}|^4 \mathrm{d}v_g)^{\frac{1}{2}}$,

ii) if $\mathrm{Y}(\Sigma) = 0$, $|k|^2 \int_\Sigma |\hat{K}|^2 \mathrm{d}v_g \leq C|k|^{\frac{1}{2}} (\mathcal{V} - \mathcal{V}_{inf})^{\frac{1}{2}} \left(\int_\Sigma |\hat{K}|^4 \mathrm{d}v_g\right)^{\frac{1}{2}}$,

iii) if $\mathrm{Y}(\Sigma) < 0$, $|k|^2 \int_\Sigma |\hat{K}|^2 \mathrm{d}v_g \leq C \left(|k|(\mathcal{V} - \mathcal{V}_{inf}) + |k|^{\frac{1}{2}} (\mathcal{V} - \mathcal{V}_{inf})^{\frac{1}{2}} \left(\int_\Sigma |\hat{K}|^4 \mathrm{d}v_g\right)^{\frac{1}{2}}\right)$

where C is a numeric constant.

Proof. i) and ii) ($Y(\Sigma) > 0$ or $Y(\Sigma) = 0$). This is immediate from the formula

$$|k|^2 \int_\Sigma |\hat{K}|^2 \mathrm{d}v_g \leq |k|^{\frac{1}{2}} (|k|^3 Vol(\Sigma))^{\frac{1}{2}} \left(\int_\Sigma |\hat{K}|^4 \mathrm{d}v_g \right)^{\frac{1}{2}}.$$

iii) $Y(\Sigma) < 0$. Assume $k = -3$ and let g_Y be the unique Yamabe metric of constant scalar curvature $R_Y = -6$ in the conformal class of g. If $g = \phi^4 g_Y$ then ϕ is determined by

$$-\Delta_{g_Y} \phi + \frac{R_Y}{8} \phi - \frac{1}{8} \phi^{-3} |\hat{K}|^2_Y + \frac{1}{12} k^2 \phi^5 = 0, \tag{67}$$

where $\Delta = \nabla^2$. The maximum principle implies (putting the values $R_Y = -6$ and $k = -3$) that

$$6(\phi^5_{\min} - \phi_{\min}) \geq \phi^{-3}_{\min} |\hat{K}|^2_Y \geq 0,$$

which makes $\phi \geq 1$. Then observe that

$$-Y(\Sigma) \leq -R_Y \Big(\int_\Sigma 1 \ \mathrm{d}v_Y \Big)^{\frac{2}{3}},$$

where $\mathrm{d}v_Y = \mathrm{d}v_{g_Y}$. This gives

$$0 \leq 6^{\frac{3}{2}} \left(\int_\Sigma \phi^6 - 1 \ \mathrm{d}v_Y \right) \leq 6^{\frac{3}{2}} \int_\Sigma \phi^6 \mathrm{d}v_Y - (-Y(\Sigma))^{\frac{3}{2}}$$
$$= \left(\frac{2}{3} k^2 Vol(\Sigma)^{\frac{2}{3}} \right)^{\frac{3}{2}} - (-Y(\Sigma))^{\frac{3}{2}}.$$

Therefore

$$\int_\Sigma (\phi - 1)^k \mathrm{d}v_Y \leq C(\mathcal{V} - \mathcal{V}_{\mathrm{inf}}),$$

for $k = 1, \ldots, 6$. Integrating Eq. (67), we get

$$6 \int_\Sigma (\phi^5 - \phi) \mathrm{d}v_Y = \int_\Sigma \phi^{-3} |\hat{K}|^2_Y \ \mathrm{d}v_Y. \tag{68}$$

Observe that

$$\begin{aligned}\int_\Sigma \phi^{-2}|\hat{K}|_Y^2 \,\mathrm{d}v_Y &= \int_\Sigma \phi\phi^{-3}|\hat{K}|_Y^2 \,\mathrm{d}v_Y \\ &= \int_\Sigma \phi^{-3}|\hat{K}|_Y^2 \,\mathrm{d}v_Y + \int_\Sigma (\phi-1)\phi^{-3}|\hat{K}|_Y^2 \,\mathrm{d}v_Y\end{aligned}$$

and

$$\begin{aligned}\int_\Sigma (\phi-1)\phi^{-3}|\hat{K}|_Y^2 \,\mathrm{d}v_Y &= \int_\Sigma (\phi-1)\phi^2\phi^{-5}|\hat{K}|_Y^2 \,\mathrm{d}v_Y \\ &\leq \left(\int_\Sigma (\phi-1)^2\phi^4 \,\mathrm{d}v_Y\right)^{\frac{1}{2}} \left(\int_\Sigma \phi^{-10}|\hat{K}|_Y^4 \,\mathrm{d}v_Y\right)^{\frac{1}{2}}. \quad (69)\end{aligned}$$

On the other hand note that

$$\left|\int_\Sigma (\phi-1)^2\phi^4 \,\mathrm{d}v_Y\right| \leq \int_\Sigma |\phi^6-1| + 2|\phi^5-1| + |\phi^4-1|\mathrm{d}v_Y \leq C(\mathcal{V}-\mathcal{V}_{\mathrm{inf}}). \quad (70)$$

Putting together Eqs. (68),(69) and (70) we get

$$\|\hat{K}\|^2_{L^2_g} \leq C\left((\mathcal{V}-\mathcal{V}_{inf}) + (\mathcal{V}-\mathcal{V}_{inf})^{\frac{1}{2}}\|\hat{K}\|^2_{L^4_g}\right), \quad (71)$$

which after scaling finishes the proof. □

Combining Propositions 8 and 9 we get

Proposition 10. *Say Σ is a compact three-manifold. Then if $Y(\Sigma) > 0$ we have*

$$\int_\Sigma 2|\nabla\hat{K}|^2 + |\hat{K}|^4 \mathrm{d}v_g \leq C(|k|\mathcal{V} + Q_0),$$

while if $Y(\Sigma) \leq 0$ we have

$$\int_\Sigma 2|\nabla\hat{K}|^2 + |\hat{K}|^4 \mathrm{d}v_g \leq C(|k|(\mathcal{V}-\mathcal{V}_{inf}) + Q_0),$$

where C is a numeric constant.

We also get

Proposition 11. *Say Σ is a compact three-manifold. If $Y(\Sigma) > 0$ we have*

$$\int_\Sigma |k|^2|\hat{K}|^2 \mathrm{d}v_g \leq C\left(|k|\mathcal{V} + (|k|\mathcal{V}Q_0)^{\frac{1}{2}}\right),$$

while if $Y(\Sigma) \leq 0$ *(same for* $Y(\Sigma) = 0$ *than for* $Y(\Sigma) < 0$*)*

$$\int_{\Sigma} |k|^2 |\hat{K}|^2 \mathrm{d}v_g \leq C \left(|k|(\mathcal{V} - \mathcal{V}_{inf}) + (|k|(\mathcal{V} - \mathcal{V}_{inf})Q_0)^{\frac{1}{2}} \right).$$

where C *is a numeric constant.*

Proof. Combine equations in Proposition 9 and Eq. (65). Making $x = |k| \|\hat{K}\|_{L^2}$ and $a = |k|^{\frac{1}{2}}(\mathcal{V} - \mathcal{V}_{inf})^{\frac{1}{2}}$ if $Y(\Sigma) \leq 0$ or $a = |k|^{\frac{1}{2}}\mathcal{V}$ if $Y(\Sigma) > 0$ we arrive at the inequality $x^2 - Cax - Ca^2 - CaQ_0^{\frac{1}{2}} \leq 0$. From it we get $x^2 \leq C(a^2 + aQ_0^{\frac{1}{2}})$. □

A direct consequence of the propositions above is

Proposition 12. *Say* Σ *is a compact three-manifold, then* $\mathcal{V}$, $|k|$ *and* Q_0 *control* $\|\widehat{\mathrm{Ric}}\|_{L^2_g}$*. In particular we have*

$$\|\widehat{\mathrm{Ric}}\|^2_{L^2_g} \leq C(|k|\mathcal{V} + Q_0),$$

where C *is a numeric constant.*

Proof. Use $\widehat{\mathrm{Ric}} = E - \frac{k}{3}\hat{K} + \hat{K} \circ \hat{K} - \frac{1}{3}|\hat{K}|^2 g$ together with the Propositions 8 and 9. □

Using the energy constraint $R = |\hat{K}|^2 - \frac{2}{3}k^2$ and Proposition 10 we get

Proposition 13. *Let* Σ *be a compact three-manifold. Then,* $\mathcal{V}$, $|k|$ *and* Q_0 *control the scalar curvature in the following way*

$$\int_{\Sigma} |\nabla R|^{\frac{4}{3}} + R^2 \mathrm{d}v_g \leq C(|k|\mathcal{V} + Q_0),$$

where C *is a numeric constant.*

Note that $|\nabla R|^{\frac{4}{3}}$ and R^2 scale as a *distance*$^{-4}$.

Proof. Squaring the energy constraint and integrating we obtain

$$\int_{\Sigma} R^2 \mathrm{d}v_g \leq \int_{\Sigma} |\hat{K}|^4 + \frac{4}{9}k^4 \mathrm{d}v_g \leq C(|k|\mathcal{V} + Q_0)$$

where in the last inequality we have used Proposition 10. On the other hand, differentiating the energy constraint we have $|\nabla R|^{\frac{4}{3}} \leq C|\nabla \hat{K}|^{\frac{4}{3}}|\hat{K}|^{\frac{4}{3}}$. Integrating and applying the Hölder inequality we obtain

$$\int_{\Sigma} |\nabla R|^{\frac{4}{3}} \mathrm{d}v_g \leq C \left(\int_{\Sigma} |\nabla \hat{K}|^2 \mathrm{d}v_g \right)^{\frac{2}{3}} \left(\int_{\Sigma} |\hat{K}|^4 \mathrm{d}v_g \right)^{\frac{1}{3}},$$

and if we apply Proposition 10 over each one of the factors on the RHS of the last equation we obtain

$$\int_{\Sigma} |\nabla R|^{\frac{4}{3}} \mathrm{d}v_g \leq C(|k|\mathcal{V} + Q_0),$$

as desired. □

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